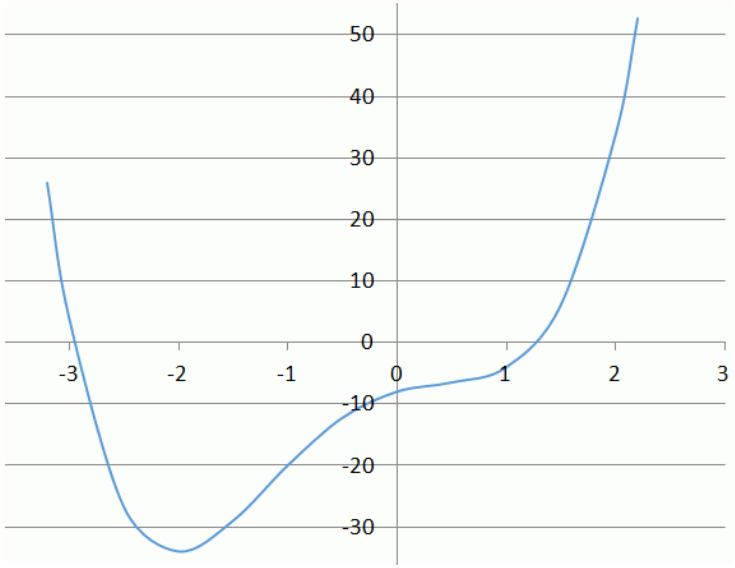
Fun Graph Problems

Alexander Lam

Graphs

- In high school, you were taught that this is a graph.



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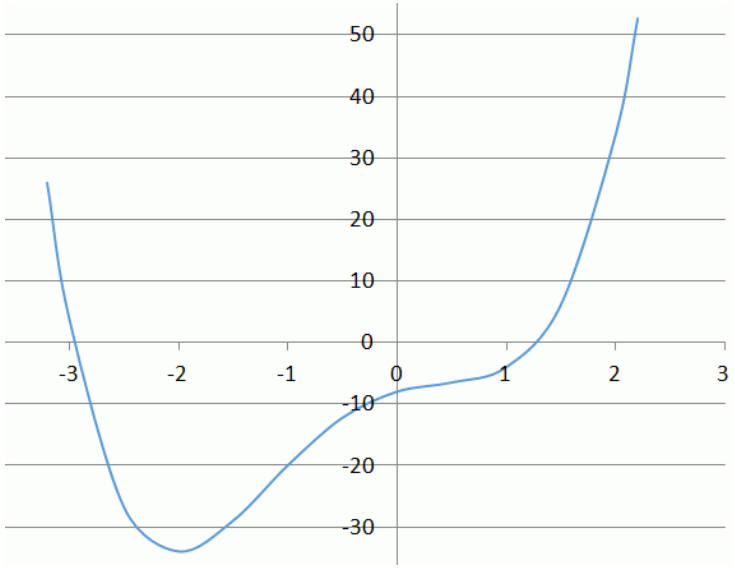
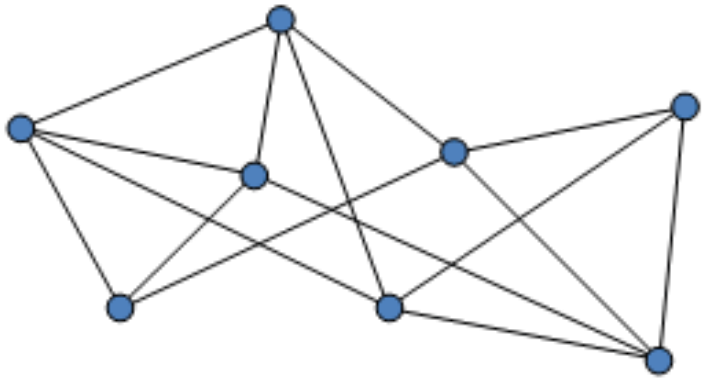


- Mathematically speaking, this is **not** a graph. It is a **plot**.

Graphs

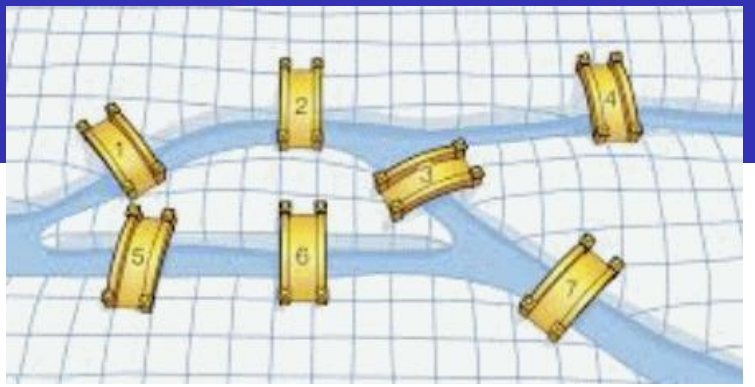
- This is a graph.



- A graph consists of a set of **vertices** (also called **nodes**), and a set of **edges.**
  - An **edge** joins two vertices together.

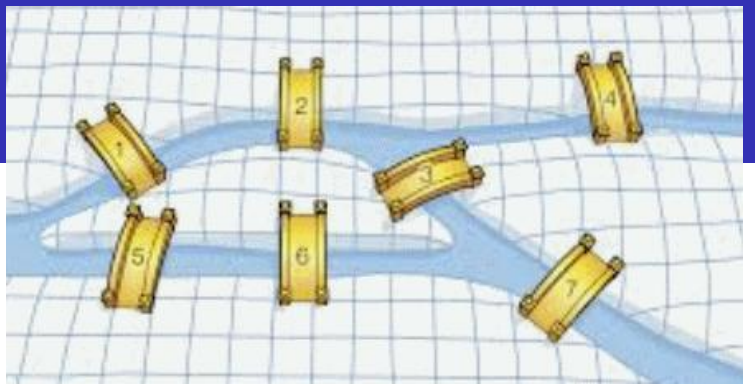
- It is one of the most useful structures for abstraction!

Seven Bridges of Königsberg



- We want to find a walk that crosses each bridge **once** and only **once**.
  - This problem was proposed by Euler in 1736

Seven Bridges of Königsberg



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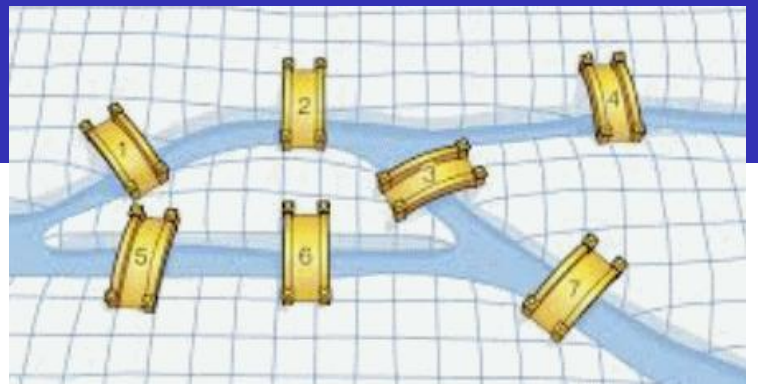
  - This problem was proposed by Euler in 1736
- Euler realised the route within the landmass was irrelevant.
- He proposed a problem abstraction which used **nodes** and **edges**.
  - In doing so, Euler laid the foundations of **graph theory,** one of the most popular fields of pure mathematics (along number theory).

Seven Bridges of Königsberg



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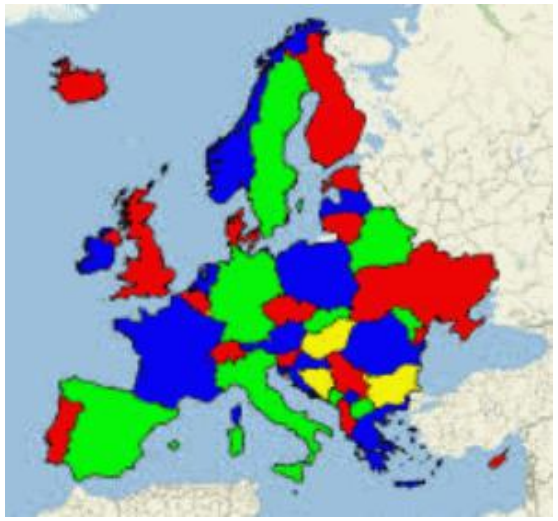
- Most of the bridges have sadly been destroyed (due to WWII bombings or highway construction).

Map Colouring Problem

- We want to colour a map such that neighbouring countries don't share a colour.



- **Graph Abstraction:** Each country is a vertex.

Map Colouring Problem

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  - This made a lot of people very angry at the time.

Noooooo!!!! You can't just use a program to check all 1,936 cases! That's not proving anything! You need to write a real proof!

Haha, computer go Brrrrrrrr

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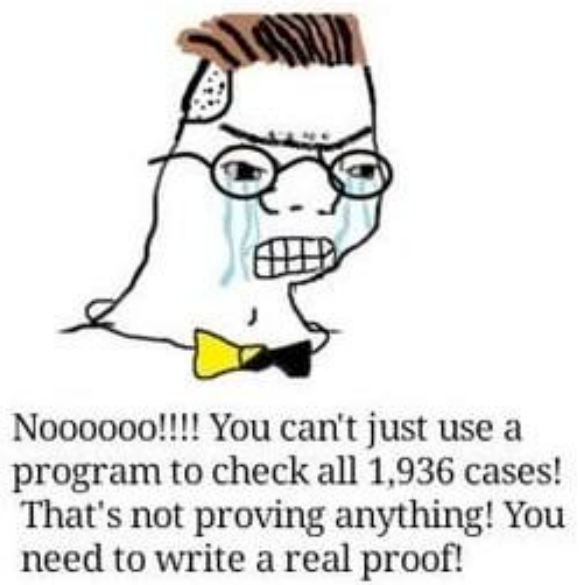
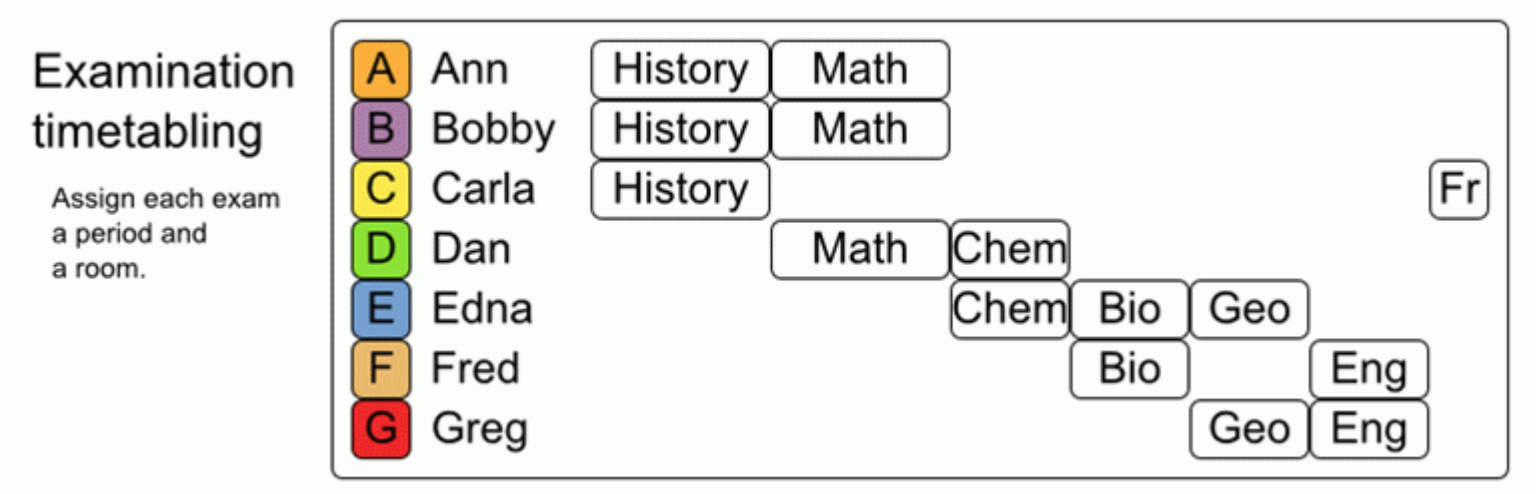


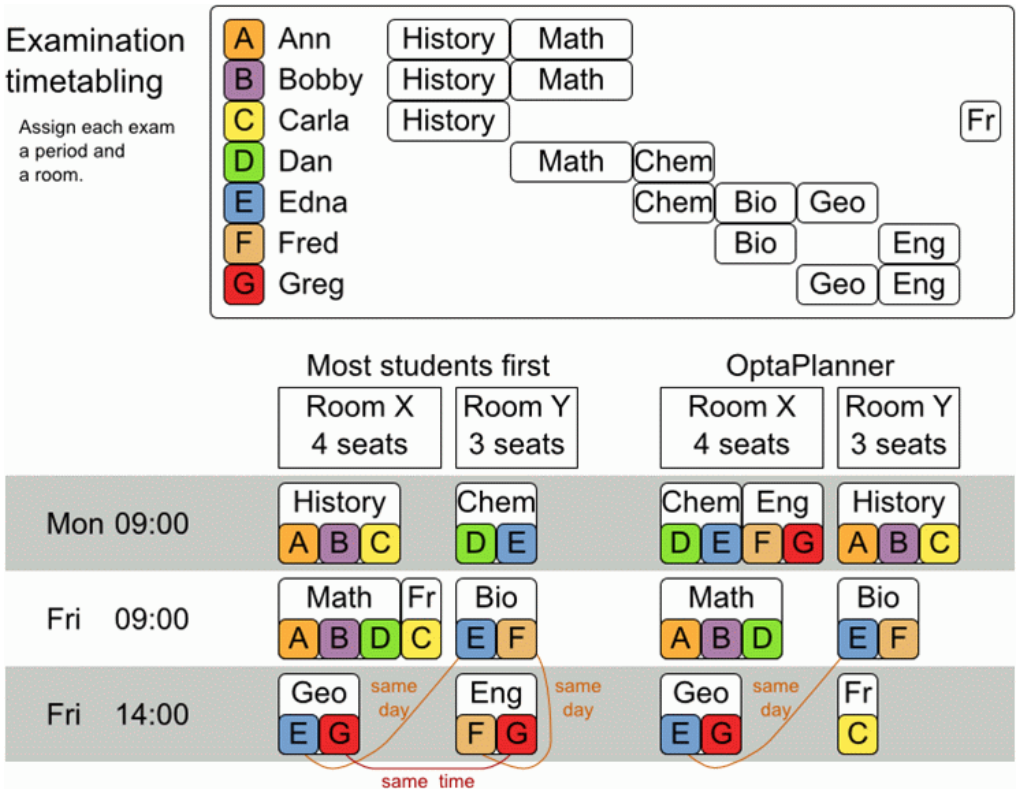
- Nowadays, computer-aided proofs are widely accepted, and I personally published a paper this year which had a computer-aided proof.

Exam Scheduling Problem

- We want to schedule all examinations to proper time slots
- We can use a graph abstraction, where each node is an exam.

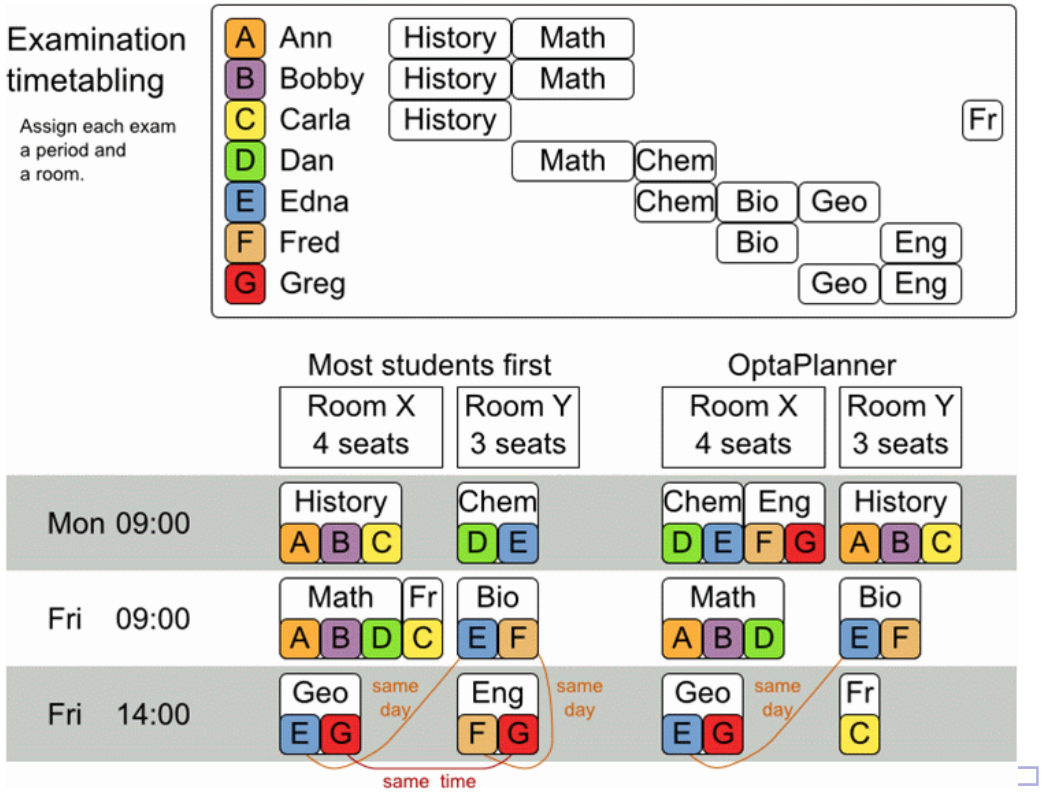


Exam Scheduling Problem



Exam Scheduling Problem

- Further considerations
  - There is a maximum capacity for each time slot.
  - There may be multiple rooms for each time slot.



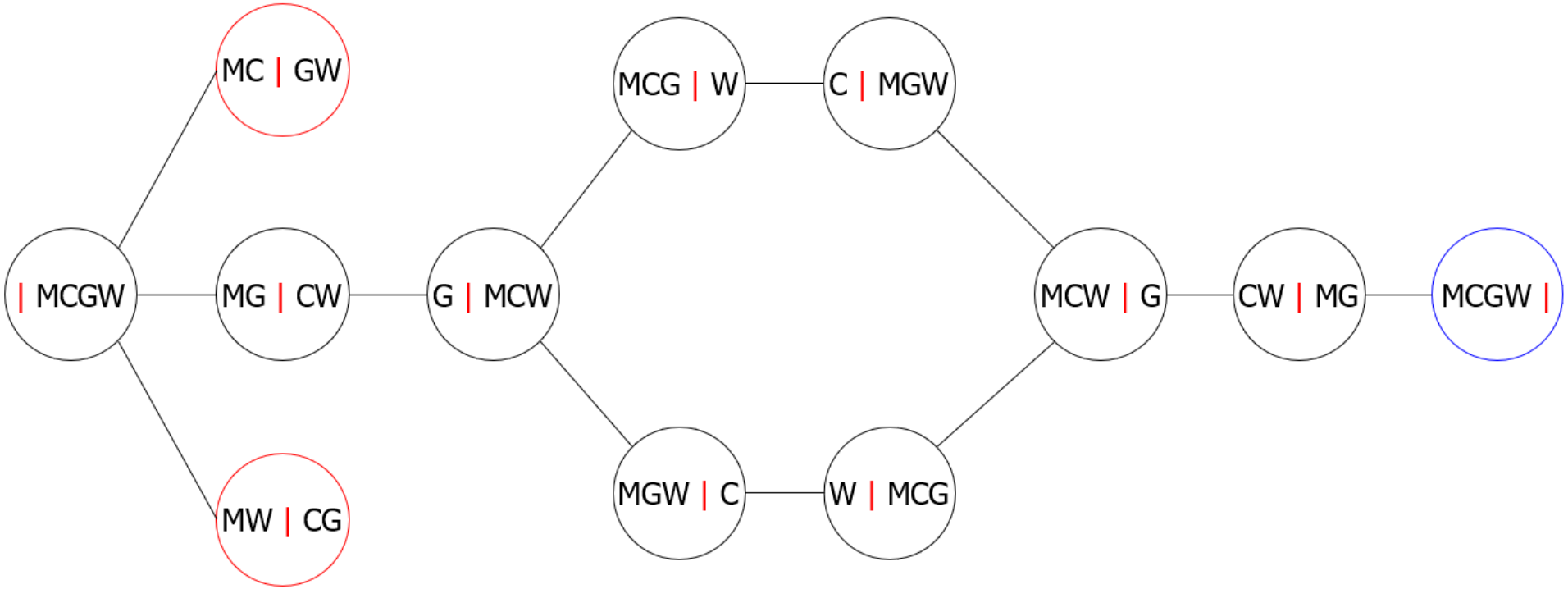
MCGW Problem

- Man, Cabbage, Goat, Wolf problem

- The man needs to bring everything across the river without anything being eaten.

MCGW Problem

- We can model our solution using a **state transition diagram**



MCGW Problem

- Another possible model.
- We assign East (E) or West (W) to indicate the current location of the man, cabbage, goat and wolf:
  - Man: E/W
  - Cabbage: E/W
  - Goat: E/W
  - Wolf: E/W

Why are we using a tuple and not a list?

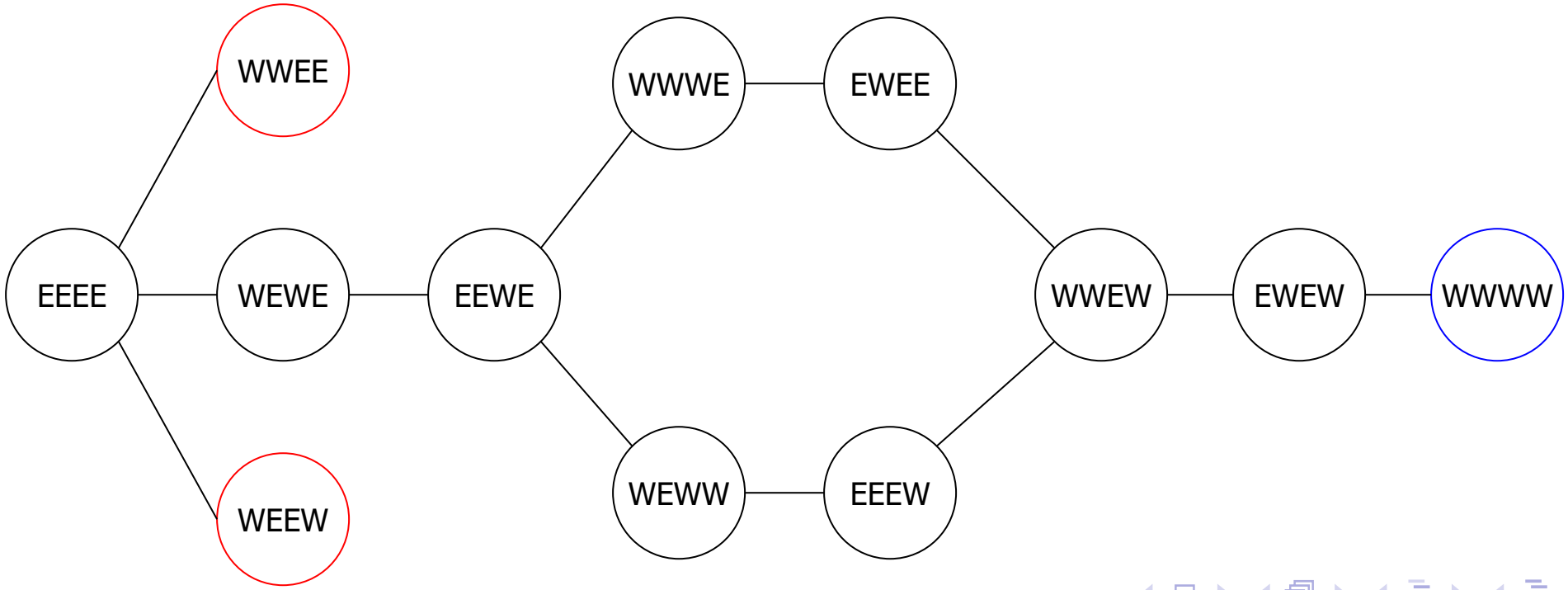
- A state is a four-tuple (E/W, E/W, E/W, E/W) for [man, cabbage, goat, wolf].
- The problem is to bring from (E, E, E, E) state to (W, W, W, W) state without losing the cabbage and goat.

MCGW Problem

- Starting from (E, E, E, E) for [man, cabbage, goat, wolf], try the three possible moves:
  - (E, E, E, E) $\Rightarrow$ (W, W, E, E)
  - (E, E, E, E) $\Rightarrow$ (W, E, W, E)
  - (E, E, E, E) $\Rightarrow$ (W, E, E, W)
- On the return trip by man:
  - (W, E, W, E) $\Rightarrow$ (E, E, W, E)
- Now we are back to the East side, repeat the steps.
  - (E, E, W, E) $\Rightarrow$ (W, W, W, E)
  - (E, E, W, E) $\Rightarrow$ (W, E, W, W)

MCGW Problem

- Another state transition diagram



MCGW Problem

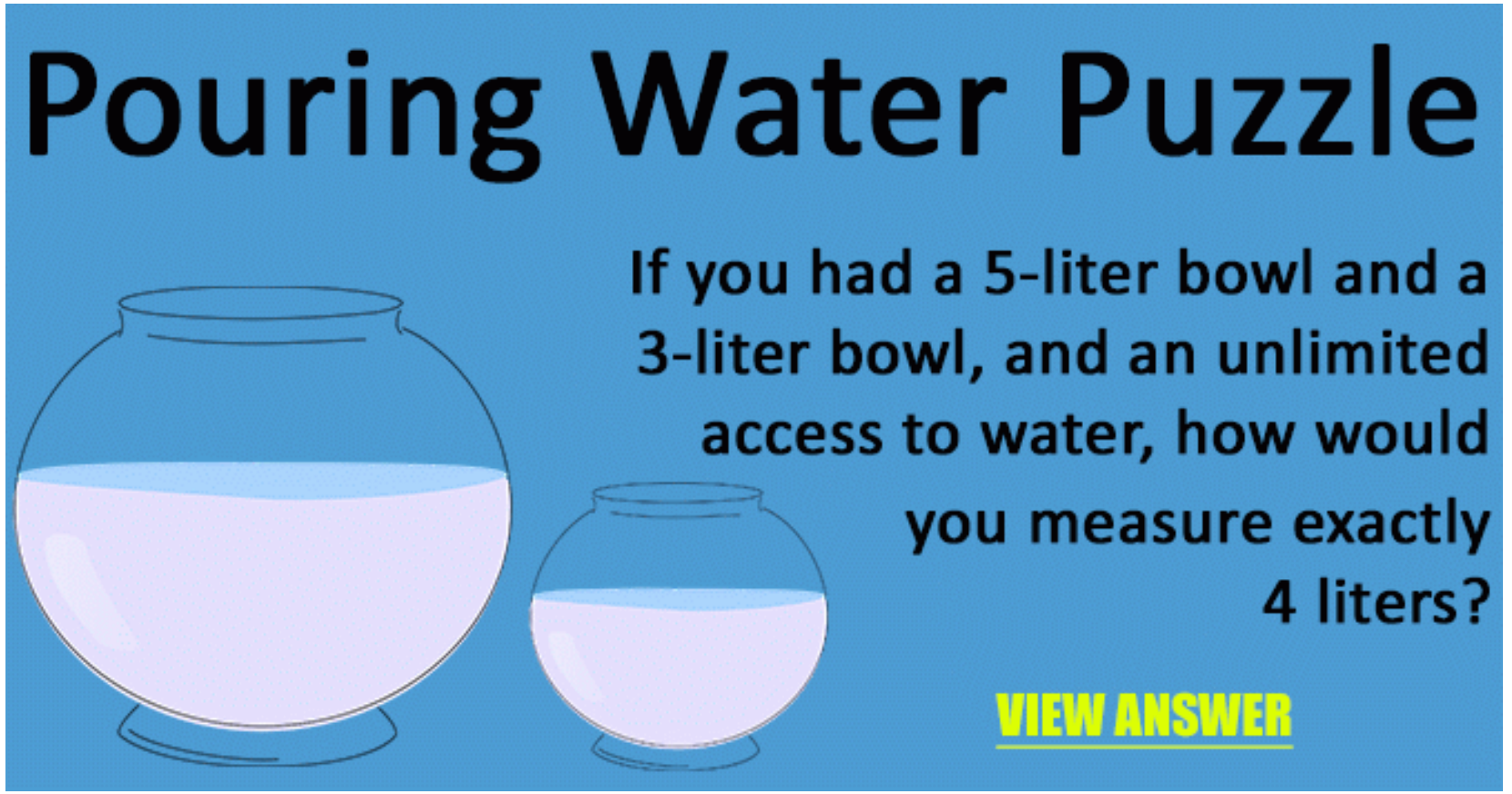
- What are the differences between the two models?
  - They are the same with the same number of nodes and links.
  - The major difference is the representation of information inside each node.
  - Each node can be represented as a 4-tuple.
- If you are to solve this problem using a computer, it is better to adopt the second model.
  - A solution is found when there is a path leading from the start node to the end node.
  - The path tells the steps needed.

MCGW Problem

- A best solution is one that contains smallest number of steps: only 2 best solutions with 7 steps.
- An acceptable solution is one that contains no cycle (EWEW $\Rightarrow$ WWEW $\Rightarrow$ EWEW is also considered a cycle): only 2.

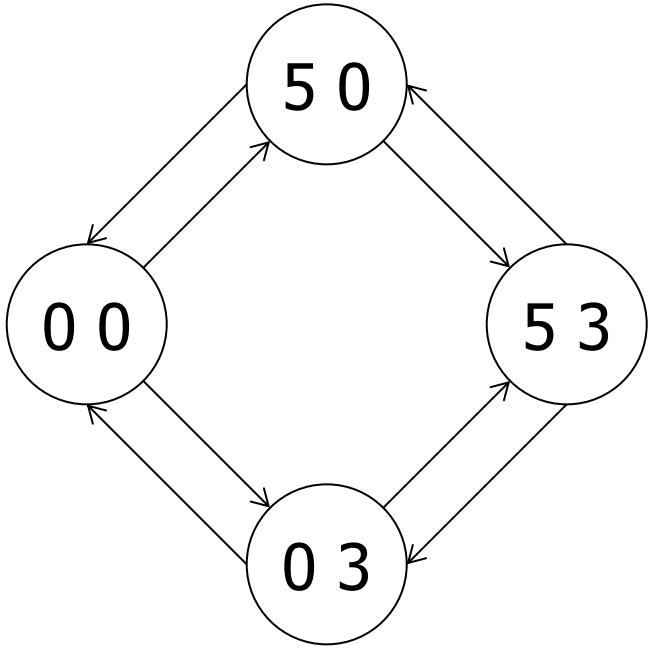
- A legal state is a state that is correct.
- An illegal state is a state that is not correct or forbidden.
  - In MCGW problem, an illegal state is a state that the wolf would eat the goat, or the goat would eat the cabbage (in the absence of the man).

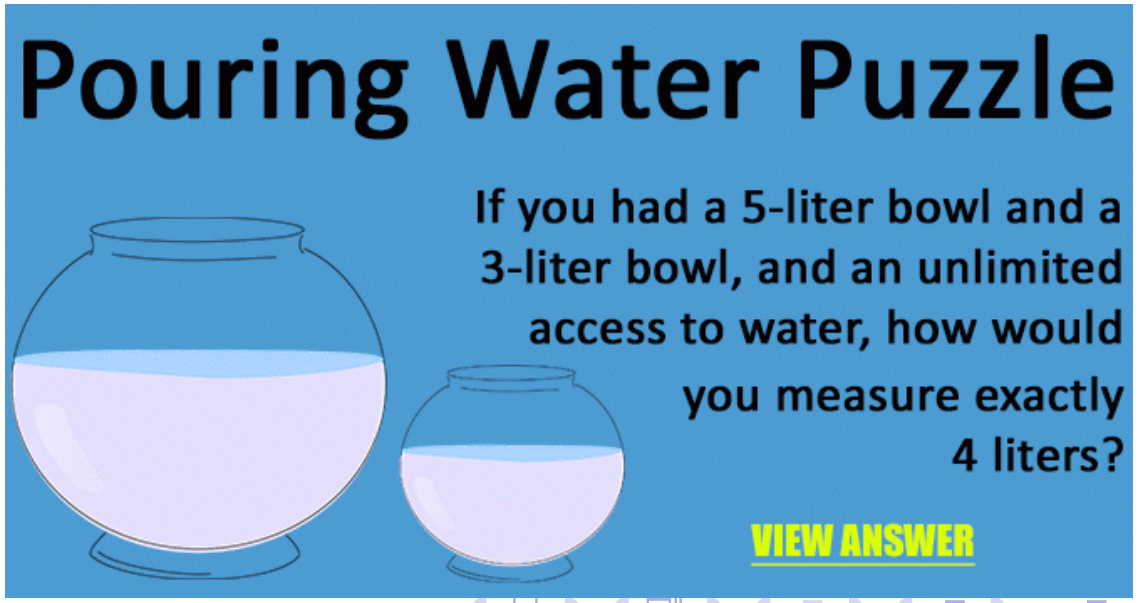
Pouring Water Problem



Pouring Water Problem

- We can use a state transition diagram to model this problem.
- Each node shows the amount of water in each bowl.
- **Activity:** Try finish the problem of measuring 4 liters.





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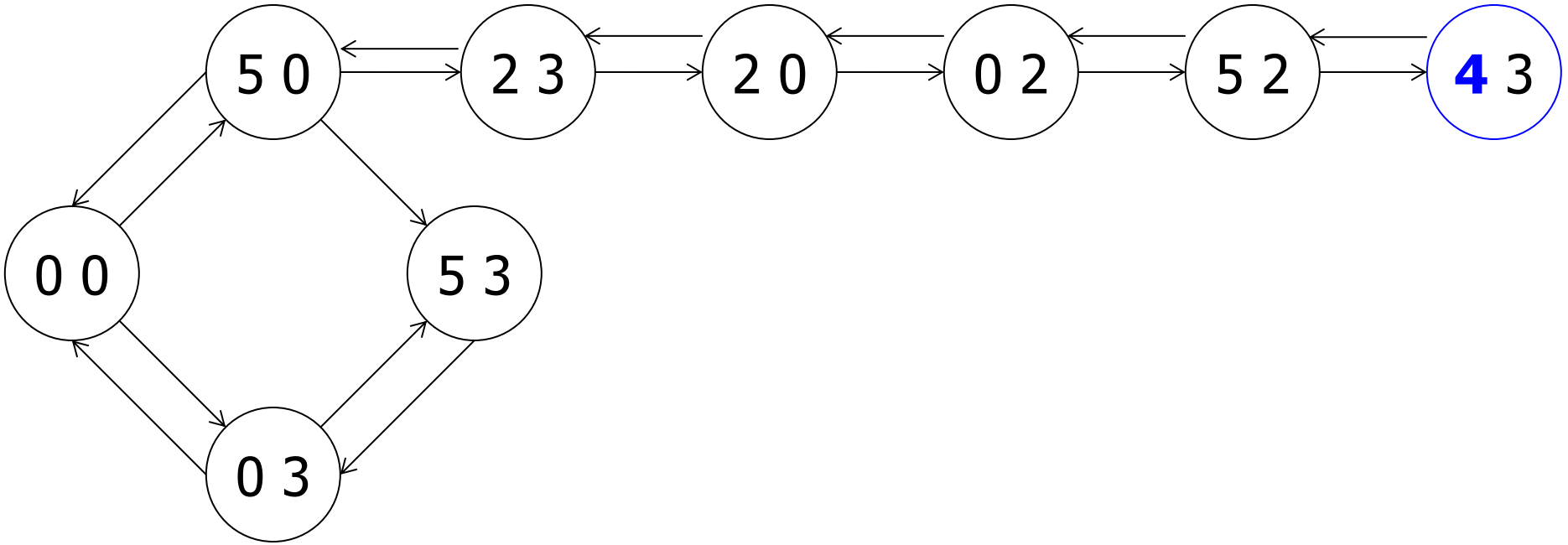
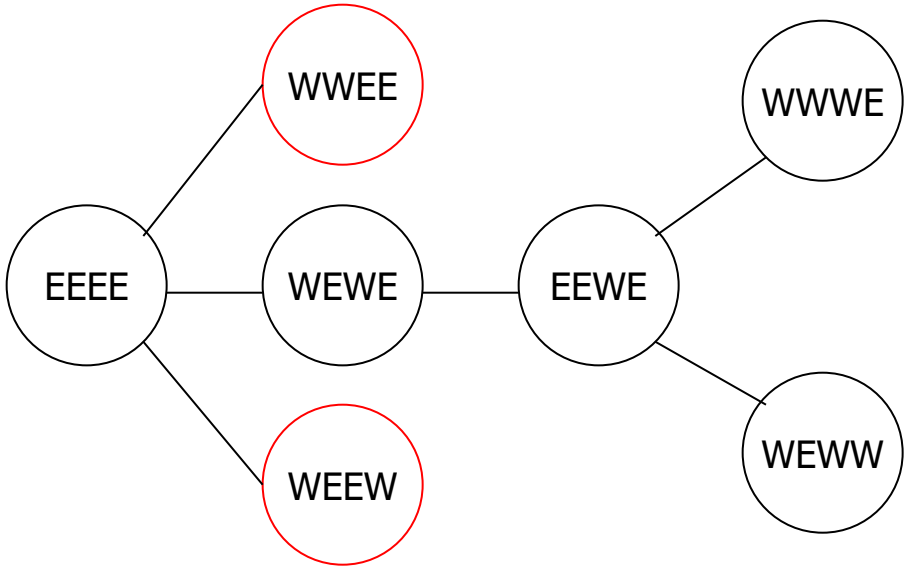


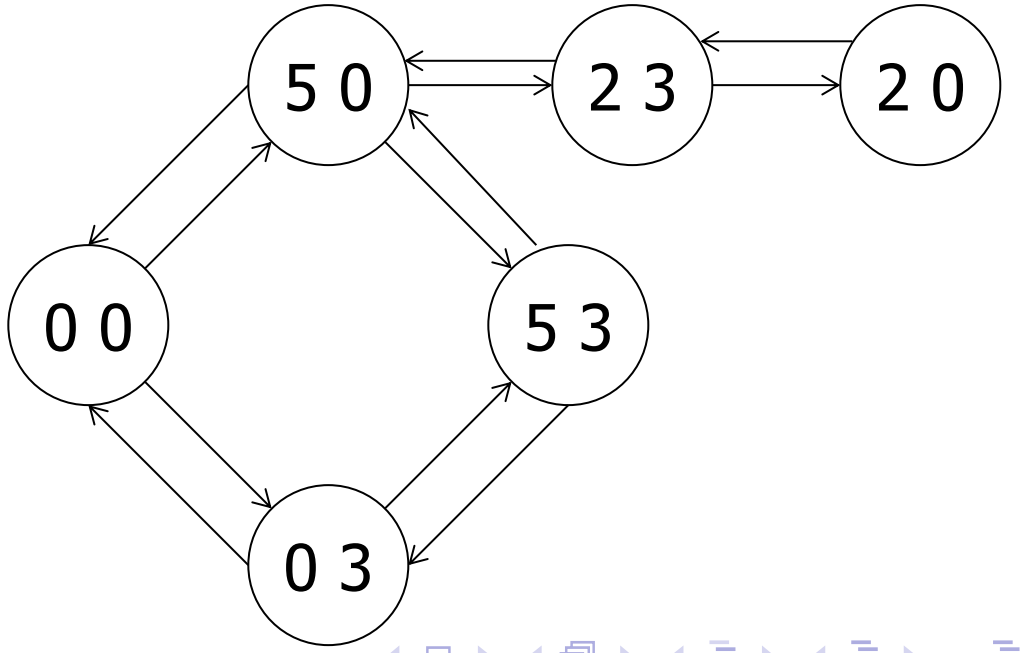
Pouring Water Problem

- If you really like this problem, you can try this at home.
- If we have two bowls of size 7 L and 11 L, can we measure out 3 L and 8 L respectively?
  - There will be more states, and far more edges.
- Try to see whether you can solve this problem.
  - How many steps are needed to measure out 3 L?
  - How many steps are needed to measure out 8 L?
    - Only 4 and 6 steps respectively.
  - How many steps are needed to measure out of 2 L and 9 L respectively?
    - 14 and 16 steps (the worst requests for 7/11 problem)

MCGW and Pouring Water

- Edges in MCGW graph have no direction.
  - Could/should there be directions?
- Edges in pouring water graph have direction.
  - Could there be no direction?





State Transition Diagrams

- State transition diagrams are a special kind of graphs.
  - States are nodes, and transitions are edges.
  - They can have directions on edges (directed graphs).

- Modeling the states of a program and the changes between states as nodes and edges of graphs is highly useful in data abstraction.
  - The hard part often is to build the proper data abstraction model.
  - You can even solve a Rubik's cube using a similar approach but it is of course much more difficult.

Recap

- We need some data representation (i.e. abstraction)
  - Helps formulate the problem to be solved by computers.
  - Sometimes, the abstraction helps us identify the key problem (e.g. the exam scheduling problem).
- We need algorithms to solve the problems.
  - An algorithm is a procedure for solving a computational problem in a finite number of steps.
    - An algorithm often involves repeating the same set of operations.
  - Some problems are too complex or too slow to solve even with correct algorithms.

Recap

- The human problem solver is responsible for the three steps of problem solving:
  - Abstracting the problem.
  - Designing an algorithm for solving the problem.
  - Coding the program to be executed.
- The computer is responsible for data crunching using the developed program.
  - For simple problems, both are easy.
  - For some problems, data abstraction is more challenging.
  - For some problems, designing the algorithm is more challenging.
  - Of course, for some problems, both are challenging.


BUT WAIT
THERE'S MORE!