Chapter 7 - GARCH models

AMA4650 Forecasting and Applied Time Series Analysis



Motivation

- Traditional econometric models assume a constant variance; e.g. ARMA.
- However, this assumption is not appropriate in many financial time series. For example, it is well known that the volatility of the stock return[1] exhibits the behavior of clustering (volatility clustering[2]).

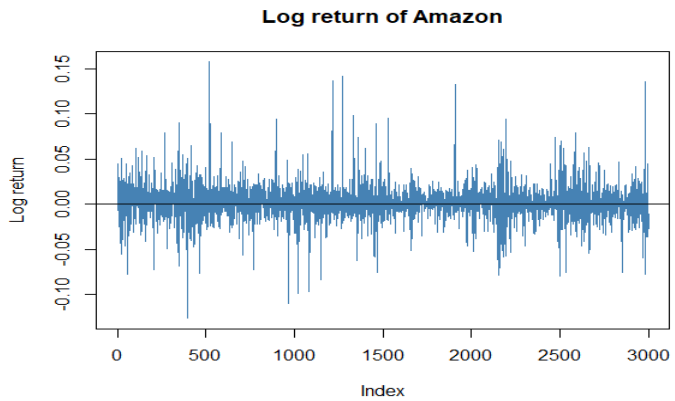


Figure: Log return of Amazon Inc.: 2010-2022.

[1] Usually, we use log-*return*; if X is the stock price, log-return is defined as $Y_t = \log(X_t) - \log(X_{t-1})$. It is also called *percentage return*.

[2] Volatility may be high for certain time periods and low for others.

Motivation, *cont'd*

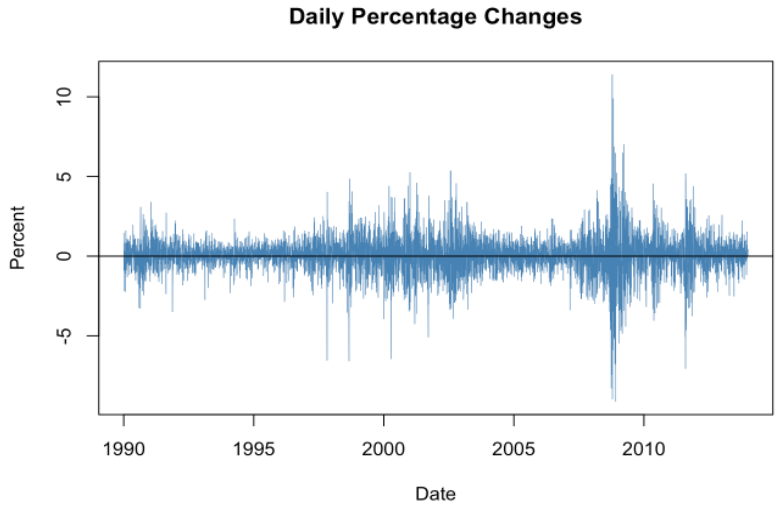


Figure: Log return of Whilshire 5000 stock index: 1989-2013.

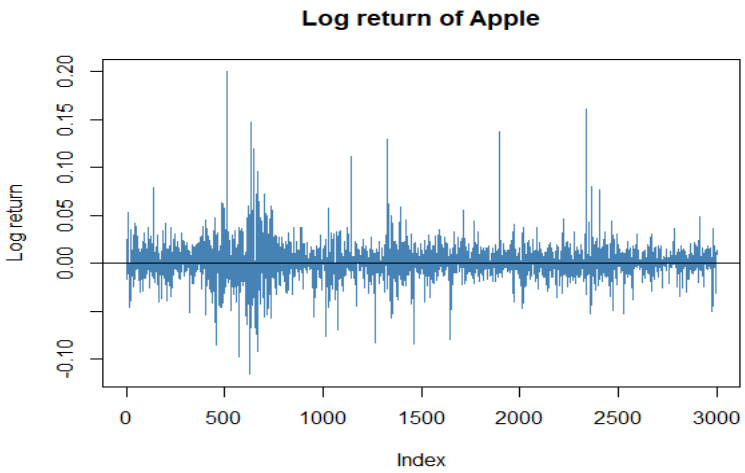


Figure: Log return of Apple Inc.: 2008-2016.

Motivation, *cont'd*

Many different time series models have been suggested for modeling time-varying volatility.

ARCH (autoregressive conditional heteroscedasticity) model: ✓

- Proposed by Engle in 1982; see

  Engle, Robert F. (1982). "*Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation*". Econometrica. 50 (4): 987–1007.

- Assume the error variance follows an AR model.
- Won 2003 Nobel Prize in economics.

GARCH (generalized ARCH) model: ✓

- Assume the error variance follows an ARMA model.

Log-normal stochastic volatility: ✗

- Assume the log-error variance follows an AR model.

...

ARCH models

- Assume the log-return of some stock follows

$$\begin{aligned} y_t &= \mu + \phi_1 x_t + \phi_2 y_{t-1} + a_t, \\ a_t &= \epsilon_t \sigma_t, \quad \epsilon_t \sim NID(0,1), \\ \sigma_t^2 &= \omega + \alpha_1 a_{t-1}^2 + \cdots + \alpha_p a_{t-p}^2. \end{aligned} \tag{1}$$

  Here $\mu, \phi_1, \phi_2, \omega, \alpha_1, \cdots, \alpha_p$ are unknown parameters, x_t is an explanatory variable, and y_{t-1} is an lagged variable of y_t. σ^2 is called volatility.

- Equation (1) is called the *conditional variance equation* or *volatility equation*.

- For simplicity, from now on, we will assume $\mu = \phi_1 = \phi_2 = 0$ ↷

ARCH models, *cont'd*

ARCH(p) model:

$$\begin{aligned} y_t &= a_t, \\ a_t &= \epsilon_t \sigma_t, \quad \epsilon_t \sim NID(0,1), \\ \sigma_t^2 &= \omega + \alpha_1 a_{t-1}^2 + \cdots + \alpha_p a_{t-p}^2, \end{aligned}$$

where $\omega > 0$ and $\alpha_i \geq 0$, $i = 1, \ldots, p$.

- The conditions $\omega > 0$ and $\alpha_i \geq 0$, $i = 1, \ldots, p$ are sufficient to ensure $\sigma_t^2 > 0$.
- Recall the information up to $t-1$ is generated by $\{a_{t-1}, a_{t-2}, \cdots\}$. Then

$$\begin{aligned} \mathbb{E}(a_t^2 | F_{t-1}) &:= \mathbb{E}(\epsilon_t^2 \sigma_t^2 | a_{t-1}, a_{t-2}, \ldots) = \sigma_t^2 \mathbb{E}(\epsilon_t^2 | a_{t-1}, a_{t-2}, \ldots) \\ &= \sigma_t^2 \mathbb{E}[\epsilon_t^2 | \epsilon_{t-1}, \epsilon_{t-2}, \cdots] = \sigma_t^2 \mathbb{E}(\epsilon_t^2) = \sigma_t^2, \end{aligned}$$

where the third equality is due to the fact that $\{a_{t-1}, a_{t-2}, \cdots\}$ and $\{\epsilon_{t-1}, \epsilon_{t-2}, \cdots\}$ are equivalent.

- The conditional mean of the error a is

$$\mathbb{E}(a_t | F_{t-1}) = \mathbb{E}(\epsilon_t \sigma_t | F_{t-1}) = \sigma_t \mathbb{E}(\epsilon_t | F_{t-1}) = 0.$$

- It implies that $\text{Var}(a_t | F_{t-1}) = \sigma_t^2$.

ARCH models, *cont'd*

ARCH(p) is stationary by assuming $0 < \alpha_1 + \cdots + \alpha_p < 1$ and $\mathbb{E}[a_0^2] < \infty$.

Proof. We only prove the result for ARCH(1): $p = 1$.

- By the property of conditional expectation, $\mathbb{E}[a_t] = \mathbb{E}[\mathbb{E}[a_t|F_{t-1}]] = 0$.
- From the conditional variance of a, we have for any k

$$\begin{aligned}\mathbb{E}[a_t^2] &= \mathbb{E}\left[\mathbb{E}[a_t^2|F_{t-1}]\right] = \mathbb{E}[\sigma_t^2] = \mathbb{E}[\omega + \alpha a_{t-1}^2] = \omega + \alpha\mathbb{E}[a_{t-1}^2] \\ &= \omega + \alpha(\omega + \alpha\mathbb{E}[a_{t-2}^2]) = \cdots = \omega + \alpha\omega + \cdots + \alpha^{k-1}\omega + \alpha^k\mathbb{E}[a_{t-k}^2] \\ &= \frac{\omega(1-\alpha^k)}{1-\alpha} + \alpha^k\mathbb{E}[a_{t-k}^2] \le \frac{\omega}{1-\alpha} + \alpha^k \sup_{s\le T}\mathbb{E}[a_s^2],\end{aligned}$$

which implies that $\sup_{t\le T}\mathbb{E}[a_t^2] \le \frac{\omega}{1-\alpha} + \alpha^k \sup_{s\le T}\mathbb{E}[a_s^2]$. Thus, $\sup_{t\le T}\mathbb{E}[a_t^2]$ is bounded. Let $k \to \infty$, we get

$$\mathbb{E}[a_t^2] = \frac{\omega}{1-\alpha}.$$

- For $k \ge 1$,

$$\begin{aligned}\text{Cov}(a_t, a_{t+k}) &= \mathbb{E}[a_t a_{t+k}] = \mathbb{E}\left\{\mathbb{E}[\sigma_t\epsilon_t\sigma_{t+k}\epsilon_{t+k}|F_{t+k-1}]\right\} \\ &= \mathbb{E}\left\{\sigma_t\epsilon_t\sigma_{t+k}\mathbb{E}[\epsilon_{t+k}|F_{t+k-1}]\right\} = 0.\end{aligned}$$

Properties of unconditional moments

Mean

Let F_{t-1} be the information set up to time $t-1$. You can imagine that it means everything related to the data up to time $t-1$ is known to the modeler. Then

$$\mathbb{E}(a_t) = \mathbb{E}\big(\mathbb{E}(a_t|F_{t-1})\big) = 0.$$

Variance

Let σ_a^2 be the variance of a_t, then

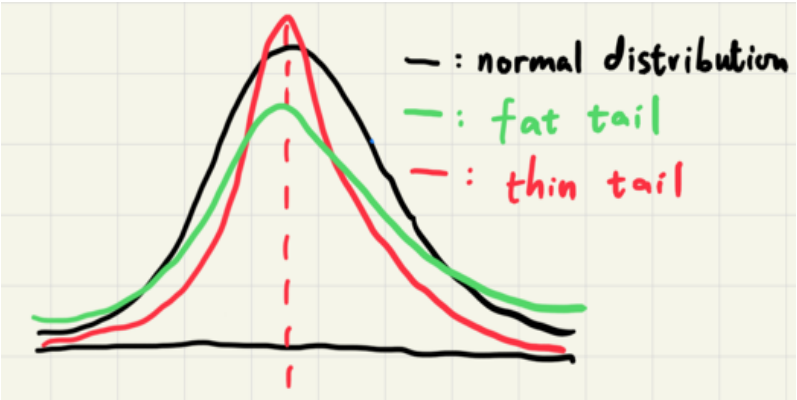
$$\begin{aligned}
\sigma_a^2 &= \mathbb{E}(a_t^2) = \mathbb{E}\big(\mathbb{E}(a_t^2|F_{t-1})\big) = \mathbb{E}(\sigma_t^2) \\
&= \mathbb{E}(\omega + \alpha_1 a_{t-1}^2 + \cdots + \alpha_p a_{t-p}^2) \\
&= \omega + \alpha_1 \mathbb{E}(a_{t-1}^2) + \cdots + \alpha_p \mathbb{E}(a_{t-p}^2) \\
&= \omega + (\alpha_1 + \cdots + \alpha_p)\sigma_a^2.
\end{aligned}$$

Hence,

$$\sigma_a^2 = \frac{\omega}{1 - \alpha_1 - \cdots - \alpha_p}.$$

Properties of unconditional moments, *cont'd*

Kurtosis $= \frac{m_4}{m_2^2}$: a measure of "tailedness" of probability distribution.



Additional restrictions on α's are required to ensure the existence of higher-order moments. To see this, we consider ARCH(1) for simplicity.

$$
\begin{aligned}
m_4 = \mathbb{E}(a_t^4) &= \mathbb{E}\big(\mathbb{E}(a_t^4|F_{t-1})\big) = \mathbb{E}\big(\mathbb{E}(\epsilon_t^4\sigma_t^4|F_{t-1})\big) \\
&= \mathbb{E}\big(\sigma_t^4\mathbb{E}(\epsilon_t^4|F_{t-1})\big) = \mathbb{E}\big(\sigma_t^4\mathbb{E}(\epsilon_t^4)\big) = \mathbb{E}(\epsilon_t^4)\mathbb{E}(\sigma_t^4) \\
&= 3\mathbb{E}\big((\omega + \alpha_1 a_{t-1}^2)^2\big) = 3\mathbb{E}(\omega^2 + 2\alpha_1\omega a_{t-1}^2 + \alpha_1^2 a_{t-1}^4) \\
&= 3\omega^2\left(1 + \frac{2\alpha_1}{1-\alpha_1}\right) + 3\alpha_1^2 m_4.
\end{aligned}
$$

(Why $\mathbb{E}[a_t^4] = \mathbb{E}[a_{t-1}^2]$? Will see later.)

Properties of unconditional moments, *cont'd*

Solving for m_4, we get

$$m_4 = \frac{3\omega^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}.$$

Hence for the fourth moment to exist, we need $0 \leq \alpha_1^2 < 1/3$.

- The standardized kurtosis, $k = m_4/(m_2)^2 - 3$, and it can be shown $k > 0$ (exercise), which implies that a_t has thicker tails than the normal distribution $(k = 0)$.
- This fits the fact that financial time series usually have thick tails.

AR representation

- Setting $\nu_t = a_t^2 - \sigma_t^2$, then it can be shown that ν_t is white noise. That is, $\mathbb{E}(\nu_t) = 0$ and $\text{Cov}(\nu_t, \nu_s) = 0$ if $t \neq s$.

  Indeed,

$$\mathbb{E}[\nu_t] = \mathbb{E}[\sigma_t^2\epsilon_t^2 - \sigma_t^2] = \mathbb{E}\{\mathbb{E}[\sigma_t^2(\epsilon_t^2 - 1)|F_{t-1}]\} = \mathbb{E}[\sigma_t^2\mathbb{E}[\epsilon_t^2 - 1|F_{t-1}]]$$
$$= \mathbb{E}[\sigma_t^2\mathbb{E}[\epsilon_t^2 - 1]] = 0.$$

  Moreover, assume $s \leq t-1 < t$, then

$$\text{Cov}(\nu_s, \nu_t) = \mathbb{E}[\nu_s\nu_t] = \mathbb{E}[\sigma_s^2(\epsilon_s^2 - 1)\sigma_t^2(\epsilon_t^2 - 1)]$$
$$= \mathbb{E}\{\mathbb{E}\left[\sigma_s^2(\epsilon_s^2 - 1)\sigma_t^2(\epsilon_t^2 - 1)|F_{t-1}\right]\} = \mathbb{E}\{\sigma_s^2(\epsilon_s^2 - 1)\sigma_t^2\mathbb{E}[(\epsilon_t^2 - 1)|F_{t-1}]\} = 0.$$

- Putting $\sigma_t^2 = a_t^2 - \nu_t$, we obtain

$$a_t^2 = \omega + \alpha_1 a_{t-1}^2 + \cdots + \alpha_p a_{t-p}^2 + \nu_t,$$

  Hence we can summarize that if $a_t \sim \text{ARCH}(p)$ then $a_t^2 \sim \text{AR}(p)$.

Remark: 1) AR(p) can be written as $\alpha(B)a_t^2 = \omega + \nu_t$, $\alpha(y) = 1 - \alpha_1 y - \cdots - \alpha_p y^p$.
2) The condition for a_t^2 to be stationary is that all roots of $\alpha(\cdot)$ are outside the unit circle. But this always holds when $\alpha_i \geq 0$ for $i = 1, \ldots, p$ and $\alpha_1 + \cdots + \alpha_p < 1$.