Midterm Exam

- March 17 (Week 9)
- Venue: QR403
- Topics: Week 1-Week 6 (including the companion tutorial in Week 7)
- Closed book with a calculator

4.2 Autoregressive processes

Why AR models?

- *Wold's decomposition theorem*. Every stationary time series Y can be written as infinite moving average

$$Y_t = \mu + \sum_{i=0}^{\infty} \psi_i \epsilon_{t-i},$$

where μ is deterministic, $\epsilon \sim WN(0, \sigma^2)$ and $\sum_{i=0}^{\infty} \psi_i^2 < \infty$.

- It is impractical to attempt to estimate infinitely many weights $\{\psi_i\}$.
- Thus, the infinite moving average representation of stationary TS is useless in practice, except for certain special cases:
  - **Case 1** - finite order MA models: only finitely many $\{\psi_i\} \neq 0$.
  - **Case 2** - finite order AR models: $\{\psi_i\}$ are generated by finite parameters: exponential decay and AR(1) ↷
  - **Case 3** - Mixture of finite order AR and MA models (ARMA).

Exponential decay of ψ_i and AR(1)

- Postulate that ψ_i has an exponential decay with rate ϕ, i.e. $\psi_i = \phi^i$.
- Wold's decomposition tells (assume $\mu = 0$ for simplicity)

$$
\begin{aligned}
Y_t &= \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} \\
\Rightarrow Y_{t-1} &= \sum_{i=0}^{\infty} \phi^i \epsilon_{t-1-i} = \sum_{j=1}^{\infty} \phi^{j-1} \epsilon_{t-j} \\
&= \frac{1}{\phi} \sum_{j=1}^{\infty} \phi^j \epsilon_{t-j} \\
&= \frac{1}{\phi} (Y_t - \epsilon_t) \\
\Rightarrow Y_t &= \phi Y_{t-1} + \epsilon_t.
\end{aligned}
$$

Autoregressive process

Definition 4.3 (The AR(p) process)

The process $\{X_t, t \in \mathbb{Z}\}$ is said to be an autoregressive process of order p, AR(p), if

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t, \quad \text{where } Z_t \sim WN(0, \sigma^2).$$

Think of an AR model as the "history repeats itself" model

- "Auto" = Self (it relies on its own past values).
- "Regressive" = Regression (predicting based on past observations).

So, an AR model is a model where the current value depends on previous values plus some randomness.

Example. If price today is based on the last two days, we might model it as:

$$P_t = 0.7P_{t-1} + 0.2P_{t-2} + \epsilon_t.$$

Example: AR(1) (1/3)

$$X_t = \phi X_{t-1} + Z_t, \quad Z \sim WN(0, \sigma^2).$$

- MA(∞) representation, or white noise representation:

$$\begin{aligned} X_t &= \phi X_{t-1} + Z_t \\ &= \phi(\phi X_{t-2} + Z_{t-1}) + Z_t = \phi^2 X_{t-2} + \phi Z_{t-1} + Z_t = \\ &= \phi^2(\phi X_{t-3} + Z_{t-2}) + \phi Z_{t-1} + Z_t \\ &= \cdots = \sum_{j=0}^{\infty} \phi^j Z_{t-j}. \end{aligned}$$

- X_t only depends on Z_j, $j \leq t$!

Example: AR(1) (2/3)

Theorem 4.2

AR(1) process $X_t = \phi X_{t-1} + Z_t$ is stationary if and only if $|\phi| < 1$.

Example: AR(1) (3/3)

Remark.

- AR process is not always stationary $\Rightarrow$ causality.
- If AR(1) is stationary, the ACF follows

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k, \quad k = 0, \pm 1, \cdots$$

- ACFs of AR(p) does not necessarily have cut off!
  $\Rightarrow$ Yule-Walker equations

ACF of AR(p): Yule-Walker equations (1/3)

- Consider the general AR(m) defined by

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdots + \alpha_m X_{t-m} + Z_t, \tag{1}$$

  or $\Phi(B)X_t = Z_t$ where $\Phi(y) = 1 - \alpha_1 y - \cdots - \alpha_m y^m$.
- Assume stationarity and $\alpha_1 + \cdots + \alpha_m \neq 1$, then $\mathbb{E}[X_t] = 0$.

Multiplying (1) by X_{t-k} and taking expectations we get

$$\gamma(k) = \alpha_1 \gamma(k-1) + \alpha_2 \gamma(k-2) + \cdots + \alpha_m \gamma(k-m), \quad k = 1, 2, \ldots.$$

- Divide both side by $\gamma(0)$:

$$\rho(k) = \alpha_1 \rho(k-1) + \alpha_2 \rho(k-2) + \cdots + \alpha_m \rho(k-m), \quad k = 1, 2, \ldots. \tag{2}$$

ACF of AR(p): Yule-Walker equations (2/3)

- Equations (2) are known as *Yule-Walker equations*. By $\rho(k) = \rho(-k)$,

$$\begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(m) \end{pmatrix} = \begin{pmatrix} 1 & \rho(1) & \cdots & \rho(m-1) \\ \rho(1) & 1 & \cdots & \rho(m-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(m-1) & \rho(m-2) & \cdots & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}.$$

and $\rho(m+j) = \alpha_1\rho(m+j-1) + \cdots + \alpha_m\rho(j)$, $j \geq 1$.

ACF of AR(p): Yule-Walker equations (3/3)

Application of Yule-Walker equations.

1. Given $(\alpha_1, \cdots, \alpha_m)$, we can solve $\{\rho(1), \cdots, \rho(m)\}$, and thus all $\{\rho(k)\}$.
2. Given $\{\rho(1), \cdots, \rho(m)\}$, we can solve $\{\alpha_1, \cdots, \alpha_m\}$. In particular,

$$\boldsymbol{\beta}_m = \mathbf{R}_m^{-1} \boldsymbol{\rho}_m$$

if $\mathbf{R}_m^{-1}$ exists, where

$$\boldsymbol{\beta}_m = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}, \boldsymbol{R}_m = \begin{pmatrix} 1 & \rho(1) & \cdots & \rho(m-1) \\ \rho(1) & 1 & \cdots & \rho(m-2) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(m-1) & \rho(m-2) & \cdots & 1 \end{pmatrix}, \boldsymbol{\rho}_m = \begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(m) \end{pmatrix}$$

From $\{\alpha_1, \cdots, \alpha_m\}$ one can construct an AR(m) model.

Stationarity/Causality of AR (1/2)

Definition 4.4

Let $\Phi(y) = 1 - \phi_1 y - \cdots - \phi_p y^p$. The $AR(p)$ process $\Phi(B)X_t = Z_t$ is said to be *causal*, if it can be rewritten as

$$X_t = \sum_{j=0}^{\infty} \theta_j Z_{t-j} = \Theta(B) Z_t,$$

where $\theta_0 = 1$, $\Theta(y) = \sum_{j=0}^{\infty} \theta_j y^j$ such that $\sum_{j=0}^{\infty} |\theta_j| < \infty$.

Remark 1. A causal AR process can be written as an MA(∞); white noise *rep*.

Remark 2. Causality implies stationarity.

Stationarity/Causality of AR (2/2)

Theorem 4.3

The AR(p) process $\Phi(B)X_t = Z_t$ is *causal* if and only if all the roots of the polynomial Φ lie *outside* the unit circle.

Proof. Compeltely the same as that of Theorem 4.1. ■

Remark 3. To get Yule-Walker equations, we assumed *stationarity* and $\phi_1 + \cdots + \phi_p \neq 1$. *Causality* implies both.

Example 4.4

Consider an AR(2) process

$$X_t = 0.7X_{t-1} - 0.1X_{t-2} + Z_t, \quad \{Z_t\} \sim WN(0, \sigma^2).$$

Verify the stationarity.

Determination of the coefficients in WN representation

- Let X be a causal AR(p) process. Then it can be written as

$$X_t = \sum_{j=0}^{\infty} \theta_j Z_{t-j}.$$

- **Determine ψ_j:** $\Phi(B)X_t = Z_t \Leftrightarrow X_t = \Theta(B)Z_t$

Example 4.4 *cont'd*

Consider an AR(2) process

$$X_t = 0.7X_{t-1} - 0.1X_{t-2} + Z_t, \quad \{Z_t\} \sim WN(0, \sigma^2).$$

$$(1 - 0.7B + 0.1B^2)X_t = Z_t \quad \Leftrightarrow \quad X_t = \Psi(B)Z_t.$$

Determine the coefficients in Ψ.

Solution:

$$1 = \Psi(B)\Phi(B)$$

$$\Leftrightarrow \quad 1 = (\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)(1 - 0.7B + 0.1B^2)$$

$$\Leftrightarrow \quad \begin{cases} 1 = \psi_0; \\ 0 = \psi_1 - 0.7\psi_0; \\ 0 = \psi_2 - 0.7\psi_1 + 0.1\psi_0; \\ 0 = \psi_3 - 0.7\psi_2 + 0.1\psi_1; \\ \vdots \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \psi_0 = 1; \\ \psi_1 - 0.7\psi_0 = 0; \\ \psi_j - 0.7\psi_{j-1} + 0.1\psi_{j-2} = 0, \quad \text{for } j \geq 2. \end{cases}$$

Example 4.4, *cont'd*

Hence,

$$\psi_0 = 1 \text{ and } \psi_1 - 0.7\psi_0 = 0 \;\Rightarrow\; \psi_1 = 0.7,$$
$$\psi_2 - 0.7\psi_1 + 0.1\psi_0 = 0 \;\Rightarrow\; \psi_2 = 0.39,$$
$$\psi_3 - 0.7\psi_2 + 0.1\psi_1 = 0 \;\Rightarrow\; \psi_3 = 0.203, \quad \cdots$$

Finally we have

$$X_t = Z_t + 0.7Z_{t-1} + 0.39Z_{t-2} + 0.203Z_{t-3} + \cdots .$$

Two methods to solve $\psi_j - 0.7\psi_{j-1} + 0.1\psi_{j-2} = 0$, for $j > 1$, in general:

- use the theory of linear difference equations (later!)
- decompose $\frac{1}{1-0.7B+0.1B^2}$ (see Tutorial)