Topic 6 - Partial Differential Equations

AMA3724 Further Mathematical Methods
Lecturer: Jianbo Cui [2024/25 Semester 1]

- Partial Differential Equations
- Regular Sturm-Liouville Problem
- Fourier series
- Heat Equation
- Wave Equation
- Laplace Equation
- Fourier transform / Inverse Fourier transform
- Black-Scholes Equation

Partial Differential Equations

- A Partial Differential Equation (PDE) is an equation containing partial derivatives of an unknown function of two or more independent variables.
- The order of a PDE is the order of the highest partial derivative in the equation.
- A second order linear partial differential equation in two independent variables x and y is an equation of the form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

  where $A = A(x,y)$, $B = B(x,y)$, $C = C(x,y)$, $D = D(x,y)$, $E = E(x,y)$, $F = F(x,y)$, and $G = G(x,y)$ are functions of x and y.
- If $G = 0$, the linear PDE is said to be homogeneous; otherwise it is non-homogeneous.
- The PDE is called $\begin{cases} \text{hyperbolic} & \text{if } B^2 - 4AC > 0, \\ \text{parabolic} & \text{if } B^2 - 4AC = 0, \\ \text{elliptic} & \text{if } B^2 - 4AC < 0. \end{cases}$

Partial Differential Equations

- **Heat Equation** [conduction of heat flow]:

$$u_t = c^2 u_{xx}$$

- **Wave Equation** [vibration of a string]:

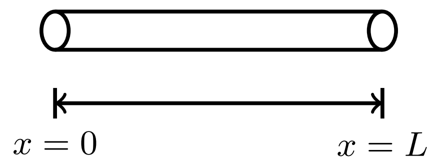
$$u_{tt} = c^2 u_{xx}$$

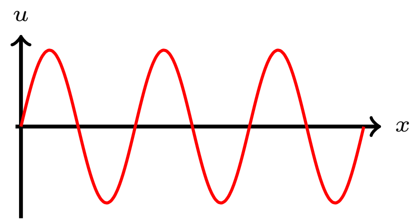
- **Laplace Equation** [electrostatic potential]:

$$u_{xx} + u_{yy} = 0$$

- **Example** Initial Boundary Value Problem (IBVP):

$u(x,t)$: Temperature at position x and time t





$$\begin{cases} u_t = c^2 u_{xx} & 0 < x < L,\ 0 < t \\ u(0,t) = 0,\ u(L,t) = 0 & 0 < t \\ u(x,0) = f(x) & 0 \le x \le L \end{cases}$$

[Boundary conditions]

[Initial condition]

Eigenvalue and eigenfunction

Given a real number λ. Consider the following second order boundary value problem.

$$\frac{d^2y}{dx^2} + \lambda y = 0 \quad \text{with} \quad y(0) = 0 \text{ and } y(L) = 0.$$

Case 1 $\lambda = 0$. Then

$$\frac{d^2y}{dx^2} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = \beta \quad \Longrightarrow \quad y = \beta x + \alpha.$$

Next, $y(0) = 0$ and $y(L) = 0$ imply

$$\begin{cases} \alpha = 0 \\ \beta L + \alpha = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}.$$

Thus, the solution is $y(x) = 0$.

Case 2 $\lambda < 0$. Then The characteristic equation is

$$\mu^2 + \lambda = 0 \quad \Longrightarrow \quad \mu = \pm\sqrt{-\lambda} \quad \Longrightarrow \quad y = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x}.$$

Next, $y(0) = 0$ and $y(L) = 0$ imply

$$\begin{cases} \alpha + \beta = 0 \\ e^{\sqrt{-\lambda}L}\alpha + e^{-\sqrt{-\lambda}L}\beta = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}.$$

Thus, the solution is $y(x) = 0$.

Eigenvalue and eigenfunction

Case 3 $\lambda > 0$. Then The characteristic equation is

$$\mu^2 + \lambda = 0 \quad \Longrightarrow \quad \mu = \pm\sqrt{\lambda}i \quad \Longrightarrow \quad y = \alpha\cos\sqrt{\lambda}x + \beta\sin\sqrt{\lambda}x.$$

Next, $y(0) = 0$ and $y(L) = 0$ imply

$$\begin{cases} \alpha = 0 \\ \alpha\cos\sqrt{\lambda}L + \beta\sin\sqrt{\lambda}L = 0 \end{cases} \Longrightarrow \begin{cases} \alpha = 0 \\ \beta\sin\sqrt{\lambda}L = 0 \end{cases}.$$

Notice that

$$\sin\sqrt{\lambda}L = 0 \quad \Longleftrightarrow \quad \sqrt{\lambda}L = n\pi \quad \Longleftrightarrow \quad \sqrt{\lambda} = \frac{n\pi}{L}.$$

The solution is $y(x) = 0$ or $y(x) = \beta\sin\frac{n\pi x}{L}$ for positive integer n. In summary,

$\lambda = 0$	$\lambda < 0$	$\lambda > 0$ and $\lambda \neq \left(\frac{n\pi}{L}\right)^2$	$\lambda > 0$ and $\lambda = \left(\frac{n\pi}{L}\right)^2$
$y(x) \equiv 0$	$y(x) \equiv 0$	$y(x) \equiv 0$	$y_n(x) = \sin\frac{n\pi x}{L}$

Here $\lambda = \left(\frac{n\pi}{L}\right)^2$ for some non-negative integer n, is called the eigenvalue of the boundary value problem and $y_n(x) = \sin\frac{n\pi x}{L}$ is called the corresponding eigenfunction.

Eigenvalue and eigenfunction

[1]:
```
from sympy import *
x,L,t,n = symbols('x L t n')
y = Function('y')(x)
```

[2]:
```
y = Function('y')(x)
DE = Eq(y.diff(x,x) + t*y,0); DE
```

[2]: $ty(x) + \frac{d^2}{dx^2}y(x) = 0$

[3]:
```
y = Function('y')
dsolve(DE,y(x),ics = {y(0):0,y(L):0})
```

[3]: $y(x) = 0$

[4]:
```
y = Function('y')(x)
t = (pi/L)**2
DE = Eq(y.diff(x,x) + t*y,0); DE
```

[4]: $\frac{d^2}{dx^2}y(x) + \frac{\pi^2 y(x)}{L^2} = 0$

[5]:
```
y = Function('y')
dsolve(DE,y(x),ics = {y(0):0,y(L):0})
```

[5]: $y(x) = C_2 e^{\frac{i\pi x}{L}} - C_2 e^{-\frac{i\pi x}{L}}$

Eigenvalue and eigenfunction

Consider the same ODE with different boundary conditions.

$$\frac{d^2y}{dx^2} + \lambda y = 0 \quad \text{with} \quad y'(0) = 0 \text{ and } y'(L) = 0.$$

Case 1 $\lambda = 0$. The general solution is $y = \beta x + \alpha$ and $y'(x) = \beta$. Then $y'(0) = 0$ and $y'(L) = 0$ imply $\beta = 0$. Thus, the solution is $y(x) = \alpha$.

Case 2 $\lambda < 0$. The general solution is $y = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x}$. Next, $y'(0) = 0$ and $y'(L) = 0$ imply

$$\begin{cases} \sqrt{-\lambda}\alpha - \sqrt{-\lambda}\beta = 0 \\ \sqrt{-\lambda}e^{\sqrt{-\lambda}L}\alpha - \sqrt{-\lambda}e^{-\sqrt{-\lambda}L}\beta = 0 \end{cases} \implies \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}.$$

Thus, the solution is $y(x) = 0$.

Case 3 $\lambda > 0$. The general solution is $y = \alpha \cos\sqrt{\lambda}x + \beta \sin\sqrt{\lambda}x$. Next, $y'(0) = 0$ and $y'(L) = 0$ imply

$$\begin{cases} \beta = 0 \\ -\alpha \sin\sqrt{\lambda}L + \beta\cos\sqrt{\lambda}L = 0 \end{cases} \implies \begin{cases} \beta = 0 \\ \alpha \sin\sqrt{\lambda}L = 0 \end{cases}.$$

Thus, the solution is $y(x) = 0$ or $y(x) = \alpha \cos\frac{n\pi x}{L}$ for integer n.

Eigenvalue and eigenfunction

In summary,

$\lambda = 0$	$\lambda < 0$	$\lambda > 0$ and $\lambda \neq \left(\frac{n\pi}{L}\right)^2$	$\lambda > 0$ and $\lambda = \left(\frac{n\pi}{L}\right)^2$
$y(x) \equiv \alpha$	$y(x) \equiv 0$	$y(x) \equiv 0$	$y_n(x) = \cos\frac{n\pi x}{L}$

Here $\lambda = \left(\frac{n\pi}{L}\right)^2$ for some non-negative integer n is the eigenvalue of the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \quad \text{with} \quad y'(0) = 0 \text{ and } y'(L) = 0,$$

and $y_n(x) = \cos\frac{n\pi x}{L}$ is called the corresponding eigenfunction.

Eigenvalue and eigenfunction

[6]:
```
DE = Eq(y.diff(x,x) + t*y,0); DE
```

[6]: $ty(x) + \frac{d^2}{dx^2}y(x) = 0$

[7]:
```
y = Function('y')
dsolve(DE,y(x),ics = {y(x).diff(x).subs(x,0):0,y(x).diff(x).subs(x,L):0})
```

[7]: $y(x) = 0$

[8]:
```
y = Function('y')(x)
t = 0
DE = Eq(y.diff(x,x) + t*y,0); DE
```

[8]: $\frac{d^2}{dx^2}y(x) = 0$

[9]:
```
y = Function('y')
dsolve(DE,y(x),ics = {y(x).diff(x).subs(x,0):0,y(x).diff(x).subs(x,L):0})
```

[9]: $y(x) = C_1$

[10]:
```
y = Function('y')(x)
t = (pi/L)**2
DE = Eq(y.diff(x,x) + t*y,0); DE
```

[10]: $\frac{d^2}{dx^2}y(x) + \frac{\pi^2 y(x)}{L^2} = 0$

[11]:
```
y = Function('y')
dsolve(DE,y(x),ics = {y(x).diff(x).subs(x,0):0,y(x).diff(x).subs(x,L):0})
```

[11]: $y(x) = C_2 e^{\frac{i\pi x}{L}} + C_2 e^{-\frac{i\pi x}{L}}$

Regular Sturm-Liouville Problem

Given the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < L \quad \text{with} \quad \begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0, \\ \beta_1 y(L) + \beta_2 y'(L) = 0. \end{cases}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants and $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$.

- The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ of the above problem are real and can be ordered such that
$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \quad \text{and} \quad \lambda_n \to \infty \text{ as } n \to \infty.$$
- For each eigenvalue λ_n, there is a non-trivial solution $y_n(x)$ which has exactly $n-1$ zeros in $(0, L)$. These eigenfunctions $y_n(x)$ satisfy
$$\int_0^L y_m(x) y_n(x)\, dx = 0 \quad \text{for all } m \neq n.$$
- If $f(x)$ and $f'(x)$ are piecewise continuous on $[0, L]$, then
$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \quad \Longrightarrow \quad c_n = \frac{\int_0^L f(x) y_n(x)\, dx}{\int_0^L y_n^2(x)\, dx}.$$

Regular Sturm-Liouville Problem

For the Regular Sturm-Liouville Problem

$$(p(x)y')' + q(x)y + \lambda w(x)y = 0, \quad a < x < b \quad \text{with} \quad \begin{cases} \alpha_1 y(a) + \alpha_2 y'(a) = 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0. \end{cases}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants and $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$, $p(x) > 0$ and $w(x) > 0$ for all $x \in [a, b]$, and $p(x)$, $q(x)$, $w(x)$, and $p'(x)$ are continuous function on $[a, b]$.

- The eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ of the regular Sturm-Liouville Problem are real and can be ordered such that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \quad \text{and} \quad \lambda_n \to \infty \text{ as } n \to \infty.$$

- For each eigenvalue λ_n, there is a non-trivial solution $y_n(x)$ which has exactly $n - 1$ zeros in (a, b). These eigenfunctions $y_n(x)$ satisfy

$$\int_a^b y_m(x) y_n(x) w(x)\, dx = 0 \quad \text{for all } m \neq n.$$

- If $f(x)$ and $f'(x)$ are piecewise continuous on $[a, b]$, then

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \quad \Longrightarrow \quad c_n = \frac{\int_a^b f(x) y_n(x) w(x)\, dx}{\int_a^b y_n^2(x) w(x)\, dx}.$$

Regular Sturm-Liouville Problem

- The boundary value problem

$$y''(x) + \lambda y(x) = 0 \quad \text{with} \quad y(0) = 0 \text{ and } y(L) = 0$$

has eigenvalues λ_n and eigenfunctions $y_n(x)$ with

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad y_n(x) = \sin\frac{n\pi x}{L} \quad n = 0, 1, 2, 3, \ldots.$$

- The boundary value problem

$$y''(x) + \lambda y(x) = 0 \quad \text{with} \quad y'(0) = 0 \text{ and } y'(L) = 0$$

has eigenvalues λ_n and eigenfunctions $y_n(x)$ with

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad y_n(x) = \cos\frac{n\pi x}{L} \quad n = 0, 1, 2, 3, \ldots.$$

- The boundary value problem

$$y''(x) + \lambda y(x) = 0 \quad \text{with} \quad y(L) = y(-L) \text{ and } y'(L) = y'(-L)$$

has eigenvalues λ_n and eigenfunctions $y_n(x)$ with

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{and} \quad y_n(x) = a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \quad n = 0, 1, 2, 3, \ldots.$$

Regular Sturm-Liouville Problem

- When $y_n(x) = \sin\frac{n\pi x}{L}$

$$\int_0^L y_m(x)y_n(x)\,dx = \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0 \quad \text{whenever } m \neq n.$$

- When $y_n(x) = \cos\frac{n\pi x}{L}$

$$\int_0^L y_m(x)y_n(x)\,dx = \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \quad \text{whenever } m \neq n.$$

- Furthermore, when $m \neq n$,

$$\begin{aligned}
\int_{-L}^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx &= 0 \quad \text{whenever } m \neq n,\\
\int_{-L}^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx &= 0 \quad \text{whenever } m \neq n,\\
\int_{-L}^L \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx &= 0 \quad \text{whenever } m \neq n.
\end{aligned}$$

Fourier cosine / sine series

Fourier cosine / sine series

Suppose f and f' are piecewise continuous on the interval $[0, L]$.

1. The **Fourier cosine series** of f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \text{for all } x \in [0, L],$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n \geq 1.$$

2. The **Fourier sine series** of f is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{for all } x \in [0, L],$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n \geq 1.$$

Fourier series

Suppose f and f' are piecewise continuous on the interval $[-L, L]$. Then f has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^{L} f(x)\, dx; \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \quad n \geq 1; \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \quad n \geq 1. \end{aligned}$$

Here a_0, a_n, and b_n are called the Fourier coefficients of $f(x)$.

Example 6.1 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_t = c^2 u_{xx} & 0 < x < L,\ 0 < t \\ u(0,t) = 0,\ u(L,t) = 0 & 0 < t \quad \text{[(Dirichlet) Boundary conditions]} \\ u(x,0) = f(x) & 0 \leq x \leq L \quad \text{[Initial condition]} \end{cases}$$

Solution. Assume a particular solution has the form

$$u(x,t) = X(x)T(t).$$

Then

$$X(x)T'(t) = u_t = c^2 u_{xx} = c^2 X''(x)T(t).$$

Thus,

$$\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is some constant. Also

$$\begin{aligned} & u(0,t) = 0 \text{ and } u(L,t) = 0 \quad t > 0 \\ \Longrightarrow\ & X(0)T(t) = 0 \text{ and } X(L)T(t) = 0 \quad t > 0 \\ \Longrightarrow\ & X(0) = 0 \text{ and } X(L) = 0. \end{aligned}$$

It follows that

$$\begin{cases} T'(t) = -\lambda c^2 T(t) \\ X''(x) + \lambda X(x) = 0 \quad \text{with } X(0) = 0 \text{ and } X(L) = 0 \end{cases}$$

The second equation is a boundary value problem with eigenvalue $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and eigenfunction $X_n(x) = \sin\frac{n\pi x}{L}$.

Next, for the first equation $T'(t) = -\lambda_n c^2 T(t)$, which is a separable ODE, the corresponding solution is

$$T_n(t) = b_n e^{-\lambda_n c^2 t} = b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t},$$

and hence

$$u_n(x,t) = X_n(x)T_n(t) = b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \sin\frac{n\pi x}{L} \quad n = 0, 1, 2, 3, \ldots$$

By the principle of superposition, the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \sin\frac{n\pi x}{L}.$$

Finally, by the initial condition $u(x,0) = f(x)$,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \Longrightarrow \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

due to the result of Fourier Sine Series. Therefore, the solution of the IBVP is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \sin\frac{n\pi x}{L} \quad \text{with} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Example 6.2 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_t = 8u_{xx} & 0 < x < 2\pi,\ 0 < t \\ u(0,t) = 0,\ u(2\pi,t) = 0 & 0 < t \qquad \text{[Boundary conditions]} \\ u(x,0) = \sin 3x & 0 \le x \le 2\pi \qquad \text{[Initial condition]} \end{cases}$$

Solution. Assume a particular solution has the form

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{T'(t)}{8T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

and hence

$$\begin{cases} T'(t) = -8\lambda T(t), \\ X''(x) + \lambda X(x) = 0 \quad \text{with } X(0) = 0 \text{ and } X(2\pi) = 0. \end{cases}$$

The eigenvalue of the second equation is $\lambda_n = \left(\frac{n}{2}\right)^2$ with eigenfunction $X_n(x) = \sin\frac{nx}{2}$.

Next, the first equation becomes

$$T'(t) = -2n^2T(t) \quad \Longrightarrow \quad T_n(t) = b_n e^{-2n^2t}.$$

By the principle of superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} b_n e^{-2n^2t}\sin\frac{nx}{2}.$$

In particular,

$$\sin 3x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{nx}{2}.$$

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By Fourier sine series expansion,

$$
\begin{aligned}
b_n &= \frac{2}{2\pi}\int_0^{2\pi} \sin 3x \sin\frac{nx}{2}\,dx \\
&= -\frac{1}{2\pi}\int_0^{2\pi} \cos\left(\frac{(n+6)x}{2}\right) - \cos\left(\frac{(n-6)x}{2}\right)dx \\
&= \begin{cases} -\dfrac{1}{2\pi}\left[\dfrac{2}{n+6}\sin\left(\dfrac{(n+6)x}{2}\right) - \dfrac{2}{n-6}\sin\left(\dfrac{(n-6)x}{2}\right)\right]_0^{2\pi} & \text{if } n \neq 6 \\ -\dfrac{1}{2\pi}\left[\dfrac{2}{n+6}\sin\left(\dfrac{(n+6)x}{2}\right) - x\right]_0^{2\pi} & \text{if } n = 6 \end{cases} \\
&= \begin{cases} 0 & \text{if } n \neq 6 \\ 1 & \text{if } n = 6 \end{cases}
\end{aligned}
$$

So the solution of IBVP is

$$u(x,t) = e^{-72t}\sin 3x.$$

Example 6.3 Solve the following Initial Boundary Value Problem (IBVP):

$$
\begin{cases} u_t = c^2 u_{xx} & 0 < x < L,\ 0 < t \\ u_x(0,t) = 0,\ u_x(L,t) = 0 & 0 < t \quad \text{[(Neumann) Boundary conditions]} \\ u(x,0) = f(x) & 0 \le x \le L \quad \text{[Initial condition]} \end{cases}
$$

Solution. Assume a particular solution has the form

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

and hence

$$
\begin{cases} T'(t) = -c^2\lambda T(t), \\ X''(x) + \lambda X(x) = 0 \quad \text{with } X'(0) = 0 \text{ and } X'(L) = 0. \end{cases}
$$

The eigenvalue of the second equation is $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with eigenfunction $X_n(x) = \cos\frac{n\pi x}{L}$. Also the general solution of the first equation is

$$T_n(t) = a_n e^{-\lambda_n c^2 t} = a_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}.$$

Then

$$u_n(x,t) = X_n(x)T_n(t) = a_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \cos\frac{n\pi x}{L} \quad n = 0, 1, 2, 3, \ldots$$

By the principle of superposition, the general solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi c}{L}\right)t} \cos\frac{n\pi x}{L}.$$

Finally, by the initial condition $u(x,0) = f(x)$ and the Fourier cosine series expansion,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \Longrightarrow \quad \begin{cases} a_0 = \dfrac{1}{L}\displaystyle\int_0^L f(x)dx \\ a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\dfrac{n\pi x}{L}dx \end{cases}.$$

Therefore, the solution of the IBVP is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \cos\frac{n\pi x}{L} \quad \text{with} \quad \begin{cases} a_0 = \dfrac{1}{L}\displaystyle\int_0^L f(x)dx, \\ a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\dfrac{n\pi x}{L}dx. \end{cases}$$

Example 6.4 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_t = c^2 u_{xx} & -L < x < L,\ 0 < t \\ u(L,t) = u(-L,t),\ u_x(L,t) = u_x(-L,t) & 0 < t \quad \text{[(Periodic) Boundary conditions]} \\ u(x,0) = f(x) & -L \le x \le L \quad \text{[Initial condition]} \end{cases}$$

Solution. Assume a particular solution has the form

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{T'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

and hence

$$\begin{cases} T'(t) = -c^2\lambda T(t), \\ X''(x) + \lambda X(x) = 0 \quad \text{with } X(L) = X(-L) \text{ and } X'(L) = X'(-L). \end{cases}$$

The eigenvalue of the second equation is $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with eigenfunction

$$X_n(x) = a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}.$$

Heat Equation

Also the general solution of the first equation is

$$T_n(t) = k_n e^{-\lambda_n c^2 t} = k_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}.$$

By the principle of superposition, the general solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi c}{L}\right)^2 t}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

Finally, by the initial condition $u(x,0) = f(x)$ and the Fourier series expansion,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \implies \begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \end{cases}.$$

Wave Equation

Example 6.5 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < L,\ 0 < t \\ u(0,t) = 0,\ u(L,t) = 0 & 0 < t \quad \text{[Boundary conditions]} \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & 0 \le x \le L \quad \text{[Initial conditions]} \end{cases}$$

Solution. Assume a particular solution has the form

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

and hence

$$\begin{cases} T''(t) + c^2\lambda T(t) = 0, \\ X''(x) + \lambda X(x) = 0 \quad \text{with } X(0) = 0 \text{ and } X(L) = 0. \end{cases}$$

The eigenvalue of the second equation is $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with eigenfunction $X_n(x) = \sin\frac{n\pi x}{L}$.

Consider the first DE

$$T''(t) + c^2 \left(\frac{n\pi}{L}\right)^2 T(t) = 0.$$

The roots of characteristics equation $\mu^2 + \left(\frac{n\pi c}{2}\right)^2 = 0$ are $\mu = \pm\frac{n\pi c}{2}i$. Thus, the general solution is

$$T_n(t) = a_n \cos\frac{n\pi ct}{L} + b_n \sin\frac{n\pi ct}{L}.$$

By the principle of superposition, the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L} + b_n \sin\frac{n\pi ct}{L} \sin\frac{n\pi x}{L}.$$

Actually, $n = 0$ is redundant. Now by the initial conditions,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \quad \text{and} \quad g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\frac{n\pi x}{L}.$$

Therefore, for $n \geq 1$,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Example 6.6 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < 1,\ 0 < t \\ u(0,t) = 0,\ u(1,t) = 0 & 0 < t \quad \text{[Boundary conditions]} \\ u(x,0) = f(x),\ u_t(x,0) = 0 & 0 \leq x \leq 1 \quad \text{[Initial conditions]} \end{cases}$$

where $f(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2}, \\ 1 - x & \frac{1}{2} < x \leq 1. \end{cases}$

Solution. Assume a particular solution has the form

$$u(x,t) = X(x)T(t).$$

Then

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

and hence

$$\begin{cases} T''(t) + \lambda T(t) = 0, \\ X''(x) + \lambda X(x) = 0 \quad \text{with } X(0) = 0 \text{ and } X(1) = 0. \end{cases}$$

The eigenvalue of the second equation is $\lambda_n = (n\pi)^2$ with eigenfunction $X_n(x) = \sin n\pi x$.

Wave Equation

Consider the first DE

$$T''(t) + (n\pi)^2 T(t) = 0.$$

The general solution is

$$T_n(t) = a_n \cos n\pi t + b_n \sin n\pi t.$$

By the principle of superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos n\pi t \sin n\pi x + b_n \sin n\pi t \sin n\pi x.$$

By the initial conditions,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin n\pi x \quad \text{and} \quad 0 = u_t(x,0) = \sum_{n=1}^{\infty} n\pi b_n \sin n\pi x.$$

Therefore, $b_n = 0$ and

$$a_n = 2\int_0^1 f(x) \sin n\pi x\, dx = \begin{cases} 0 & n \text{ is even,} \\ \dfrac{4(-1)^{\frac{n-1}{2}}}{n^2\pi^2} & n \text{ is odd.} \end{cases}$$

i.e.,

$$u(x,t) = \sum_{n \text{ is odd}} \frac{4(-1)^{\frac{n-1}{2}}}{n^2\pi^2} \cos n\pi t \sin n\pi x.$$

Laplace Equation

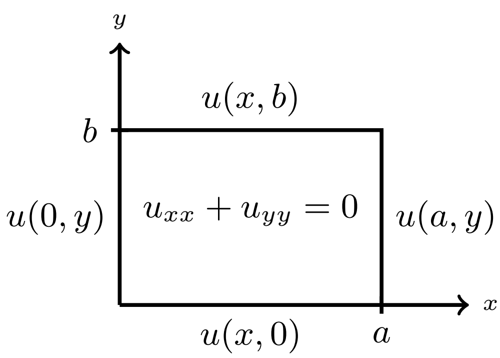
Example 6.7 Solve the following Boundary Value Problem (BVP):

$$\begin{cases} u_{xx} + u_{yy} = 0 & D = \{(x,y) : 0 < x < a, 0 < y < b\} \\ u(0,y) = 0,\ u(a,y) = 0, & 0 < y < b \\ u(x,0) = 0,\ u(x,b) = f(x) & 0 < x < a \end{cases}$$

[Boundary conditions]

Solution. Assume a particular solution has the form

$$u(x,y) = X(x)Y(y)$$



Then

$$-\frac{Y''(y)}{Y(y)} = \frac{X''(x)}{X(x)} = -\lambda,$$

and hence

$$\begin{cases} Y''(y) - \lambda Y(y) = 0, \\ X''(x) + \lambda X(x) = 0 \quad \text{with } X(0) = 0 \text{ and } X(a) = 0. \end{cases}$$

The eigenvalue of the second equation is $\lambda_n = \left(\dfrac{n\pi}{a}\right)^2$ with eigenfunction $X_n(x) = \sin \dfrac{n\pi x}{a}$.

Laplace Equation

Consider the first DE

$$Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) = 0.$$

The general solution is

$$Y_n(y) = a_n e^{\frac{n\pi y}{a}} + b_n e^{-\frac{n\pi y}{a}}.$$

By the principle of superposition, the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} a_n e^{\frac{n\pi y}{a}} \sin\frac{n\pi x}{a} + b_n e^{-\frac{n\pi y}{a}} \sin\frac{n\pi x}{a}.$$

Now by the remaining two boundary conditions,

$$\begin{aligned} 0 = u(x,0) &= \sum_{n=1}^{\infty}(a_n + b_n)\sin\frac{n\pi x}{a} \\ f(x) = u(x,b) &= \sum_{n=1}^{\infty}(a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}})\sin\frac{n\pi x}{a}. \end{aligned}$$

By considering the Fourier sine series expansion,

$$a_n = \frac{2}{a(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}})}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx \quad \text{and} \quad b_n = -a_n.$$

Non-homogeneous boundary conditions

Example 6.8 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_t = c^2 u_{xx} & 0 < x < L,\ 0 < t \\ u(0,t) = u_0,\ u(L,t) = u_L & 0 < t \qquad \text{[Boundary conditions]} \\ u(x,0) = f(x) & 0 \le x \le L \qquad \text{[Initial condition]} \end{cases}$$

Solution. Let

$$u(x,t) = v(x) + w(x,t) \quad \text{with} \quad v(x) = \left(\frac{u_L - u_0}{L}\right)x + u_0.$$

Then $u_t = w_t$ and $u_{xx} = w_{xx}$ and hence $w_t = u_t = c^2 u_{xx} = c^2 w_{xx}$. Now

$$\begin{aligned} w(0,t) &= u(0,t) - v(0) = 0, \\ w(L,t) &= u(L,t) - v(L) = 0, \\ w(x,0) &= u(x,0) - v(x) = f(x) - v(x). \end{aligned}$$

Therefore,

$$\begin{cases} w_t = c^2 w_{xx} & 0 < x < L,\ 0 < t \\ w(0,t) = 0,\ w(L,t) = 0 & 0 < t \qquad \text{[Boundary conditions]} \\ w(x,0) = f(x) - v(x) & 0 \le x \le L \qquad \text{[Initial condition]} \end{cases}$$

Example 6.9 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_t = c^2 u_{xx} & 0 < x < L,\ 0 < t \\ u(0,t) = T_0(t),\ u(L,t) = T_L(t) & 0 < t \quad \text{[Boundary conditions]} \\ u(x,0) = f(x) & 0 \le x \le L \quad \text{[Initial condition]} \end{cases}$$

Solution. Let

$$u(x,t) = v(x,t) + w(x,t) \quad \text{with} \quad v(x,t) = \left(\frac{T_L(t) - T_0(t)}{L}\right) x + T_0(t).$$

Then $u_t = v_t + w_t$ and $u_{xx} = w_{xx}$ and hence

$$\begin{cases} w_t = c^2 w_{xx} - v_t & 0 < x < L,\ 0 < t \\ w(0,t) = 0,\ w(L,t) = 0 & 0 < t \quad \text{[Boundary conditions]} \\ w(x,0) = f(x) - v(x,0) & 0 \le x \le L \quad \text{[Initial condition]} \end{cases}$$

Example 6.10 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_t = 4u_{xx} + \sin 3\pi x & 0 < x < 1,\ 0 < t \\ u(0,t) = 0,\ u(1,t) = 0 & 0 < t \quad \text{[Boundary conditions]} \\ u(x,0) = 5\sin \pi x & 0 \le x \le 1 \quad \text{[Initial condition]} \end{cases}$$

Solution. Let $u(x,t) = v(x) + w(x,t)$ such that

$$4v''(x) + \sin 3\pi x = 0 \quad \text{with} \quad v(0) = 0 \text{ and } v(1) = 0.$$

Solving the above DE, we obtain

$$v(x) = \frac{1}{(6\pi)^2} \sin 3\pi x.$$

Furthermore,

$$\begin{cases} w_t = 4w_{xx} & 0 < x < 1,\ 0 < t \\ w(0,t) = 0,\ w(1,t) = 0 & 0 < t \quad \text{[Boundary conditions]} \\ w(x,0) = 5\sin \pi x - \frac{1}{(6\pi)^2} \sin 3\pi x & 0 \le x \le 1 \quad \text{[Initial condition]} \end{cases}$$

By Example 7.1,

$$w(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2\pi^2 t} \sin n\pi x$$

with

$$\begin{aligned} b_n &= 2\int_0^1 \left(5\sin \pi x - \frac{1}{(6\pi)^2}\sin 3\pi x\right)\sin n\pi x\,dx \\ &= \begin{cases} 10\displaystyle\int_0^1 \sin^2 \pi x\,dx = 5 & n=1 \\ -\dfrac{2}{(6\pi)^2}\displaystyle\int_0^1 \sin^2 3\pi x\,dx = -\dfrac{1}{(6\pi)^2} & n=3 \\ 0 & n\neq 1,3 \end{cases} \end{aligned}$$

Therefore,

$$u(x,t) = v(x)+w(x,t) = \frac{1}{(6\pi)^2}\sin 3\pi x + 5e^{-4\pi^2 t}\sin \pi x - \frac{1}{(6\pi)^2}e^{-36\pi^2 t}\sin 3\pi x.$$

Example 6.11 Solve the following Initial Boundary Value Problem (IBVP):

$$\begin{cases} u_t = c^2 u_{xx} + g(x,t) & 0<x<L,\ 0<t \\ u(0,t)=0,\ u(L,t)=0 & 0<t \quad \text{[Boundary condition]} \\ u(x,0)=f(x) & 0\le x\le L \quad \text{[Initial condition]} \end{cases}$$

Solution. Consider the boundary value problem

$$X''(x) + \lambda X(x) = 0 \quad \text{with} \quad X(0)=0 \text{ and } X(L)=0.$$

The eigenvalue of the second equation is $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with eigenfunction $X_n(x) = \sin\frac{n\pi x}{L}$. Let

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) \quad \text{and} \quad g(x,t) = \sum_{n=1}^{\infty} g_n(t)X_n(x),$$

with

$$g_n(t) = \frac{2}{L}\int_0^L g(x,t)X_n(x)\,dx = \frac{2}{L}\int_0^L g(x,t)\sin\frac{n\pi x}{L}\,dx.$$

Non-homogeneous Heat Equation

$$\begin{aligned} u_t(x,y) &= c^2 u_{xx}(x,y) + g(x,y) \\ \sum_{n=1}^{\infty} T_n'(t) X_n(x) &= c^2 \sum_{n=1}^{\infty} T_n(t) X_n''(x) + \sum_{n=1}^{\infty} g_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} -\lambda_n c^2 T_n(t) X_n(x) + \sum_{n=1}^{\infty} g_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} \left(-c^2 \lambda_n T_n(t) + g_n(t)\right) X_n(x) \end{aligned}$$

Thus,

$$T_n'(t) = -c^2 \lambda_n T_n(t) + g_n(t) \quad \Longrightarrow \quad T_n'(t) + c^2 \lambda_n T_n(t) = g_n(t),$$

which is a linear DE. Then the general solution is

$$T_n(t) = e^{-\lambda_n c^2 t} \left(\int_0^t e^{\lambda_n c^2 y} g_n(y)\, dy + T_n(0) \right).$$

Finally,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} T_n(0) X_n(x) \quad \Longrightarrow \quad T_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

Fourier transform / Inverse Fourier transform

Fourier transform

The **Fourier transform** function $F(w)$ of the function $y = f(x)$ is defined by

$$\mathcal{F}[y] = \mathcal{F}[f(x)] = F(w) = \int_{-\infty}^{\infty} e^{-iwx} f(x)\, dx.$$

Remark: Such integral exists when f satisfies $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.

Inverse Fourier transform

The **inverse Fourier transform** of $F(w)$ is given by

$$f(x) = \mathcal{F}^{-1}[F(w)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwx} F(w)\, dw.$$

Equivalently,

$$\mathcal{F}^{-1}[F(w)] = f(x) \quad \Longleftrightarrow \quad F(x) = \mathcal{F}[f(x)].$$

Fourier transform / Inverse Fourier transform

Suppose $\mathcal{F}[f(x)] = F(w)$ and $\mathcal{F}[g(x)] = G(w)$. For any constants a and b,

- **Linearity**:
$$\mathcal{F}[af(x) + bg(x)] = aF(w) + bG(w)$$
- **Shift in the x domain**:
$$\mathcal{F}[f(x-a)] = e^{-iwa}F(w)$$
- **Shift in the w domain**:
$$\mathcal{F}[e^{iaw}f(x)] = F(w-a)$$
- **Transform of Derivatives**:
$$\mathcal{F}\left[f^{(n)}(x)\right] = (iw)^n F(w), \quad n \geq 1$$
- **Derivative of Transform**:
$$\frac{d^n}{dw^n}F(w) = \mathcal{F}\left[(-iw)^n f(x)\right], \quad n \geq 1$$
- **Convolution**:
$$\mathcal{F}^{-1}[F(w)G(w)] = f * g(x) = \int_{-\infty}^{\infty} f(s)g(x-s)\,ds$$
- **Symmetry**:
$$\mathcal{F}[F(x)] = 2\pi f(-w)$$

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Table of Fourier transform

$f(x)$	$F(w)$	$f(x)$	$F(w)$
1	$2\pi\delta(w)$	e^{iw_0x}	$2\pi\delta(w-w_0)$
$\lvert x\rvert$	$-\frac{2}{w^2}$	$e^{-\alpha\lvert x\rvert}$	$\frac{2\alpha}{w^2+\alpha^2}$
$\delta(x)$	1	$e^{-\frac{x^2}{2\sigma^2}}$	$\sqrt{2\pi}\sigma e^{-\frac{1}{2}\sigma^2w^2}$
$H(x)$	$\pi\delta(w) + \frac{1}{iw}$	$e^{-\alpha x}H(x)$	$\frac{1}{\alpha+iw}$
$\operatorname{sgn}(w)$	$\frac{2}{iw}$	$x^n e^{-\alpha x}H(x)$	$\frac{n!}{(\alpha+iw)^{n+1}}$
$\cos w_0x$	$\pi(\delta(w+w_0)+\delta(w-w_0))$	$H(x)\cos w_0x$	$\frac{\pi}{2}(\delta(w+w_0)+\delta(w-w_0)) + \frac{iw}{w_0^2-w^2}$
$\sin w_0x$	$i\pi(\delta(w+w_0)-\delta(w-w_0))$	$H(x)\sin w_0x$	$\frac{i\pi}{2}(\delta(w+w_0)-\delta(w-w_0)) + \frac{iw}{w_0^2-w^2}$

The Heaviside step function and delta function $\delta(x)$ are defined by

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \text{and} \quad \delta(x) = \lim_{a\to 0}\delta_a(x) \quad \text{with} \quad \delta_a(x) = \begin{cases} \frac{1}{a} & -\frac{a}{2} \leq x \leq \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$$

Example 6.12 Solve the following Initial Value Problem (IVP):

$$\begin{cases} u_t = c^2 u_{xx} & -\infty < x < \infty,\ 0 < t \\ u(x,0) = f(x) & -\infty < x < \infty \end{cases} \qquad \text{[Initial condition]}$$

Solution. Let $U(w,t) = \mathcal{F}[u(x,t)]$ and $\mathcal{F}[f(x)] = F(w)$.Then

$$\begin{aligned} u_t(x,t) &= c^2 u_{xx}(x,t) \\ \mathcal{F}[u_t(x,t)] &= c^2 \mathcal{F}[u_{xx}(x,t)] \\ U_t(w,t) &= c^2 (iw)^2 U(w,t) \\ U_t(w,t) &= -c^2 w^2 U(w,t) \end{aligned}$$

Then it become an ordinary differential equation with respect to t.

$$U_t(w,t) = -c^2 w^2 U(w,t) \quad \text{with} \quad U(w,0) = F(w).$$

The solution is

$$U(w,t) = F(w) e^{-c^2 w^2 t}.$$

Taking the inverse Fourier transform,

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1}[U(w,t)] &= \mathcal{F}^{-1}\left[F(w) e^{-c^2 w^2 t}\right] \\ &= \mathcal{F}^{-1}[F(w)] * \mathcal{F}^{-1}\left[e^{-c^2 w^2 t}\right] \\ &= f(x) * \left(\frac{1}{2\sqrt{c^2 \pi t}} e^{-\frac{x^2}{4c^2 t}}\right) \\ &= \frac{1}{2\sqrt{c^2 \pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4c^2 t}}\, ds \end{aligned}$$

This solution is called the fundamental solution of heat equation.

$$u(x,t) = \frac{1}{2\sqrt{c^2 \pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4c^2 t}}\, ds.$$

Example 6.13 Solve the following Initial Value Problem (IVP):

$$\begin{cases} u_{xx} + u_{yy} = 0 & -\infty < x < \infty,\ 0 < y \\ u(x,0) = f(x) & -\infty < x < \infty \qquad \text{[Initial condition]} \end{cases}$$

and $|u(x,y)|$ is bounded as $y \to \infty$.

Solution. Let $U(w,y) = \mathcal{F}[u(x,y)]$ and $\mathcal{F}[f(x)] = F(w)$.Then

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= 0 \\ \mathcal{F}[u_{xx}(x,y)] + \mathcal{F}[u_{yy}(x,y)] &= 0 \\ (iw)^2 U(w,y) + U_{yy}(w,y) &= 0 \\ U_{yy}(w,y) - w^2 U(w,y) &= 0 \end{aligned}$$

Then it become an ordinary differential equation with respect to y.

$$U_{yy}(w,y) - w^2 U(w,y) = 0 \quad \text{with} \quad U(w,0) = F(w).$$

The general solution is

$$U(w,y) = A(w)e^{|w|y} + B(w)e^{-|w|y}.$$

Since $|u(x,y)|$ is bounded as $y \to \infty$, we have $A(w) = 0$. Furthermore, $U(w,0) = F(w)$ implies $B(w) = F(w)$. Therefore,

$$U(w,y) = F(w)e^{-|w|y}.$$

Taking the inverse Fourier transform,

$$\begin{aligned} u(x,ty = \mathcal{F}^{-1}[U(w,y)] &= \mathcal{F}^{-1}\left[F(w)e^{-|w|y}\right] \\ &= \mathcal{F}^{-1}[F(w)] * \mathcal{F}^{-1}\left[e^{-|w|y}\right] \\ &= f(x) * \left(\frac{y}{\pi(x^2+y^2)}\right) \\ &= \int_{-\infty}^{\infty} f(s) \cdot \frac{y}{\pi((x-s)^2+y^2)}\, ds \end{aligned}$$

This solution is called the Poisson's Integral Formula for the half plane.

$$u(x,t) = \int_{-\infty}^{\infty} \frac{yf(s)}{\pi((x-s)^2+y^2)}\, ds.$$

- $C = C(S, t)$, the value of a call option with current value of the underlying asset S at time t;
- $P = P(S, t)$, the value of a put option with current value of the underlying asset S at time t;
- $V = V(S, t)$, the value of an option with current value of the underlying asset S at time t;
- σ, the volatility of the underlying asset;
- E, the exercise price;
- T, the expiry time;
- r, the risk-free interest rate.

At the expiry time $t = T$,

$$C(S, T) = \max\{S - E, 0\} \quad \text{and} \quad P(S, T) = \max\{E - S, 0\}.$$

The following assumptions are made.

- The asset price follows the lognormal random walk.
- The risk-free interest rate r and the asset volatility σ are known functions.
- There are no transaction costs.
- The asset pays no dividends during the life of the option.
- There are no arbitrage possibilities.
- Trading of the asset can take place continuously.
- Short selling is permitted.

Then the value of an option $V(S, t)$ satisfies the following Black-Scholes partial differential equation.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Black-Scholes Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Let

$$\begin{cases} S = Ee^x & \Longrightarrow \quad \dfrac{dS}{dx} = S \\ t = T - \dfrac{2}{\sigma^2}y & \Longrightarrow \quad \dfrac{dt}{dy} = -\dfrac{2}{\sigma^2} \end{cases}$$

Then

$$\begin{aligned} \frac{\partial V}{\partial x} &= \frac{\partial V}{\partial S}\cdot\frac{dS}{dx} = S\frac{\partial V}{\partial S} \\ \frac{\partial^2 V}{\partial x^2} &= \frac{\partial}{\partial x}\left(S\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(S\frac{\partial V}{\partial S}\right)\cdot\frac{dS}{dx} = \left(\frac{\partial V}{\partial S} + S\frac{\partial^2 V}{\partial S^2}\right)\cdot S \\ \frac{\partial V}{\partial y} &= \frac{\partial V}{\partial t}\cdot\frac{dt}{dy} = -\frac{2}{\sigma^2}\frac{\partial V}{\partial t} \end{aligned}$$

Then

$$S\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}, \quad \frac{\partial V}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial V}{\partial y} \quad \text{and} \quad S^2\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}.$$

Black-Scholes Equation

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0 \\ -\frac{\sigma^2}{2}\frac{\partial V}{\partial y} + \frac{\sigma^2}{2}\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) + r\frac{\partial V}{\partial x} - rV &= 0 \\ -\frac{\partial V}{\partial y} + \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV &= 0 \quad \text{with} \quad k = \frac{2r}{\sigma^2} \end{aligned}$$

with $k = \dfrac{2r}{\sigma^2}$. Let

$$V(x,y) = Ee^{ax+by}u(x,y) \quad \Rightarrow \quad u(x,y) = \frac{1}{E}e^{-(ax+by)}V(x,y).$$

Then

$$\begin{aligned} \frac{\partial V}{\partial x} &= Ee^{ax+by}\left(au + \frac{\partial u}{\partial x}\right) \\ \frac{\partial V}{\partial y} &= Ee^{ax+by}\left(bu + \frac{\partial u}{\partial y}\right) \\ \frac{\partial^2 V}{\partial x^2} &= Ee^{ax+by}\left(a^2u + 2a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) \end{aligned}$$

Then

$$Ee^{ax+by}\left[-\left(bu+\frac{\partial u}{\partial y}\right)+\left(a^2u+2a\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\right)+(k-1)\left(au+\frac{\partial u}{\partial x}\right)-ku\right] = 0$$

$$-\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial x^2}+(2a+k-1)\frac{\partial u}{\partial x}+(a^2+(k-1)a-k-b)u = 0$$

Set

$$a=-\frac{k-1}{2} \quad \text{and} \quad b=-\frac{(k+1)^2}{4}$$

Then we have

$$-\frac{\partial u}{\partial y}+\frac{\partial^2 u}{\partial x^2}=0 \quad \Longrightarrow \quad \frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x^2}$$

which is a Heat Equation. Notice that

$$x=\ln\frac{S}{E}, \quad y=\frac{r}{k}(T-t), \quad V(S,t)=\frac{1}{E}e^{-\frac{(k-1)}{2}\ln\frac{S}{E}-\frac{(k+1)^2r}{4k}(T-t)}\cdot u\left(\ln\frac{S}{E},\frac{r}{k}(T-t)\right).$$

Example Consider the following European call option model

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rV=0.$$

with conditions

$$C(S,T)=\max\{S-E,0\}, \quad C(S,t)\to S \text{ when } S\to\infty.$$

The model can be reformulated as

$$\begin{cases} u_y=u_{xx} & -\infty<x<\infty,\ 0<y \\ u(x,0)=u_0(x)=\max\{e^{(k+1)x/2}-e^{(k-1)x/2},0\} & -\infty<x<\infty \text{ [Initial condition]} \end{cases}$$

The general solution is given (from the previous example) by

$$u(x,y)=\frac{1}{2\sqrt{\pi y}}\int_{-\infty}^{\infty}u_0(s)e^{-\frac{(x-s)^2}{4y}}\,ds$$

and

$$C(S,t)=SN(d_1)-Ee^{-r(T-t)}N(d_2) \quad \text{with} \quad N(d)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}e^{-\frac{s^2}{2}}\,ds$$

where

$$d_1=\frac{\ln(S/E)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2=\frac{\ln(S/E)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$