

Topic 5 - Ordinary Differential Equation

AMA3724 Further Mathematical Methods(2024/25 Semester 1)
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- ▶ Separate equation
- ▶ Homogeneous equation
- ▶ Linear equation
- ▶ Exact equation
- ▶ Variation of parameters
- ▶ Method of undetermined coefficients
- ▶ Applications

Ordinary differential equation

P. 2 / 77

Continuously Compounded Interest

At a bank, **continuous compounding interest** means that interest is accrued at a rate that is fixed percentage of the balance at that moment.

Suppose a bank account earns interest continuously at a rate of 5% of the current balance and the initial deposit is \$100. Let

t = time (independent variable),

$B(t)$ = Balance in the account at time t (dependent variable).

Then

$$\frac{dB}{dt} = (0.05)B.$$

This equation is an example of an **Ordinary Differential Equation (ODE)**. With the condition $B(0) = 100$, the pair of equations

$$\frac{dB}{dt} = (0.05)B \quad \text{with} \quad B(0) = 100.$$

is called an **Initial-Value Problem (IVP)**. It can be shown that

$$B(t) = 100e^{0.05t}$$

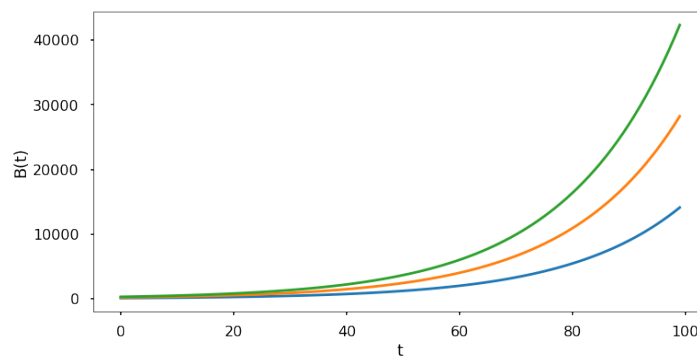
is the **solution to the IVP**.

```
[1]: import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

# define the function f(t,B)
def f(t,B):
    return 0.05 * B

# Solve the ODE numerically
sol = solve_ivp(f, [0, 100], [100,200,300], t_eval=np.arange(0,100,1))

# Plot the solution
plt.figure(figsize = (12, 6))
for y in sol.y:
    plt.plot(sol.t,y)
plt.xlabel('t')
plt.ylabel('B(t)')
plt.show()
```



First order ordinary differential equations:

1. $\frac{dy}{dx} = \frac{4y}{x(y-3)}$
2. $y' = \frac{2}{y} - \frac{1}{x^2}$
3. $x \frac{dy}{dx} - y + 3x^3y - x^4 = 0$
4. $x^2y' + 3xy = 4x^2 + 2$

Second order ordinary differential equations:

1. $\frac{d^2y}{dx^2} - 3x^2 \frac{dy}{dx} + 2y = 0$
2. $y'' - 2xy' + 2 \sin xy = \cos x.$

Second order non-homogeneous linear differential equations with constant coefficients:

1. $\frac{d^2y}{dx^2} - 3 \frac{dy}{dx} + 2y = x^2$
2. $y'' - 2y' + 2y = \cos x.$

Initial Value Problems:

1. $\frac{dy}{dx} = x^3 - 2 \sin x, \quad y(0) = 3$
2. $\frac{d^2y}{dx^2} + 4y = 3 \cos x, \quad y(0) = 0, \quad y'(0) = 0$

First order ordinary differential equation

- ▶ A **first-order differential equation** is an equation for an unknown function in terms of its derivative. The standard form for a first-order differential equation is

$$\frac{dy}{dx} = f(x, y), \quad \text{or equivalently, } y' = f(x, y).$$

- ▶ A **solution** of the differential equation is a function $y(x)$ that satisfies

$$\frac{d}{dt}(y(x)) = f(x, y(x)).$$

- ▶ A differential equation along with an initial condition is called an **initial value problem (IVP)**. The usual form of an initial value problem is

$$\frac{dy}{dx} = f(x, y) \quad \text{with } y(x_0) = y_0.$$

A first order differential equation of the form

- ▶ $\frac{dy}{dx} = f(x, y) = g(x)h(y)$ is called a **separable** equation.

- ▶ $\frac{dy}{dx} = f(x, y) = g\left(\frac{y}{x}\right)$ is called a **homogeneous** equation.

- ▶ $\frac{dy}{dx} + a(x)y = b(x)$ is called a **linear** equation.

- ▶ $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ is called an **exact** equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$$1. \frac{dy}{dx} = \frac{4y}{x(y-3)}$$

$$9. \frac{dy}{dx} = -\frac{2y-1}{2x-y+2}$$

$$2. y' = \frac{2}{y} - \frac{1}{x^2}$$

$$10. (y^2 - x^2) + (2xy + 1) \frac{dy}{dx} = 0$$

$$3. x \frac{dy}{dx} - y + 3x^3y - x^4 = 0$$

$$11. (y^2 - x^2) + (2xy + 1) \frac{dy}{dx} = 0$$

$$4. \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

$$12. \frac{dy}{dx} = \frac{x+3y}{x-y}$$

$$5. x^2y' + 3xy = 4x^2 + 2$$

$$13. \frac{dy}{dx} = \frac{x-y-1}{x+3}$$

$$6. y' = x^4y$$

$$14. \frac{dy}{dx} = \frac{-2y+1}{2x-y+2}$$

$$7. e^{-y} \frac{dy}{dx} + (x^2 + 1) = 0$$

$$15. \frac{dy}{dx} = \frac{x-3y-1}{2x-6y+3}$$

$$8. \frac{dy}{dx} = \frac{y - \sqrt{x^2 + y^2}}{x}$$

$$16. \frac{dy}{dx} = \frac{-2y+1}{3x-y+2}$$

Separable equation

Separable equation

A first order differential equation of the form

$$\frac{dy}{dx} = f(x, y) = g(x)h(y),$$

is called a **separable** equation.

General Method for Separation of Variables

If one has a separable type DE,

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \\ \frac{1}{h(y)} \frac{dy}{dx} &= g(x) \\ \underbrace{\int \frac{1}{h(y)} dy}_{H(y)} &= \underbrace{\int g(x) dx}_{G(x)} + C \end{aligned}$$

By evaluating the two indefinite integrals $H(y)$ and $G(x)$ on both sides, one can solve the separable DE. We call this method **Separation of Variables**. The **general solution** is

$$H(y) = G(x) + C.$$

Homogeneous equation

A function $f(x, y)$ is said to be **homogeneous of degree n** if

$$f(\lambda x, \lambda y) = \lambda^n f(x, y) \quad \text{for all scalar } \lambda.$$

A first order differential equation $\frac{dy}{dx} = f(x, y)$ is a **homogeneous equation** if $f(x, y) = g\left(\frac{y}{x}\right)$ is homogeneous of degree zero.

General Method for Homogeneous equation

Set $y = ux$ and hence $\frac{dy}{dx} = u + x \frac{du}{dx}$. Therefore,

$$u + x \frac{du}{dx} = g(u) \implies \frac{du}{dx} = \frac{1}{x} (g(u) - u),$$

which is a separable DE. Then the general solution is

$$\int \frac{du}{g(u) - u} = \int \frac{1}{x} dx + C$$

$$\int \frac{du}{g(u) - u} = \ln|x| + C$$

$$G\left(\frac{y}{x}\right) = \ln|x| + C \quad \text{with} \quad G(u) = \int \frac{du}{g(u) - u}.$$

Linear equation

Linear equation

A first order differential equation is **linear** if it can be written in the form

$$\frac{dy}{dx} + a(x)y = b(x),$$

where $a(x)$ and $b(x)$ are functions of x .

General Method for Linear equation

Define an **integrating factor** by

$$\mu(x) = \exp\left(\int a(x) dx\right) = e^{\int a(x) dx} \implies \frac{d}{dx}(\mu(x)) = a(x) \cdot \left(e^{\int a(x) dx}\right) = a(x) \mu(x).$$

Multiply both side of linear differential equation by $\mu(x)$,

$$\mu(x) \frac{dy}{dx} + \mu(x) a(x) y = \mu(x) b(x)$$

$$\frac{d}{dx}(\mu(x) y) = \mu(x) b(x)$$

$$\mu(x) y = \int \mu(x) b(x) dx$$

$$y = \frac{1}{\mu(x)} \left(\int \mu(x) b(x) dx + C \right).$$

Exact equation

A first order differential equation of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

is called **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Necessary condition for Exact equation: Consider an equation $f(x, y) = C$ for some C^2 function f and constant C . Differentiate the equation with respect to x , we obtain

$$\frac{\partial}{\partial x} (f(x, y)) = \frac{\partial}{\partial x} (C) \implies \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0.$$

\parallel
 $M(x, y)$

\parallel
 $N(x, y)$

$$\begin{cases} \frac{\partial f}{\partial x} = M(x, y) \\ \frac{\partial f}{\partial y} = N(x, y) \end{cases} \implies \begin{cases} \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \\ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x} \end{cases} \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{(Necessary condition)}$$

General Method for Exact equation

Suppose $M(x, y) + N(x, y) \frac{dy}{dx} = 0$ is exact. We consider

$$\begin{cases} \frac{\partial f}{\partial x} = M(x, y) \\ \frac{\partial f}{\partial y} = N(x, y) \end{cases}$$

From the first equation, we have

$$f(x, y) = \int M(x, y) dx + g(y),$$

where g is a function of y only. We then differentiate this equation with respect to y and compare with the second equation,

$$N(x, y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left(\int M(x, y) dx + g(y) \right) = \int \frac{\partial M}{\partial y} dx + g'(y).$$

Then

$$g'(y) = N(x, y) - \int \frac{\partial M}{\partial y} dx \implies g(y) = \int \left(N(x, y) - \int \frac{\partial M}{\partial y} dx \right) dy.$$

Finally, the general solution is

$$f(x, y) = \int M(x, y) dx + \int \left(N(x, y) - \int \frac{\partial M}{\partial y} dx \right) dy = C.$$

Example 5.1 Find the general solution of the following DEs.

$$1. \quad \frac{dy}{dx} = \frac{4y}{xy - 3x}$$

$$2. \quad \frac{dy}{dx} = y^2$$

Solution. The DE is separable.

$$\begin{aligned} \frac{dy}{dx} &= \frac{4y}{xy - 3x} \\ \frac{dy}{dx} &= \frac{4}{x} \cdot \frac{y}{y - 3} \\ \int \frac{y - 3}{y} dy &= \int \frac{4}{x} dx \\ y - 3 \ln |y| &= 4 \ln |x| + C. \end{aligned}$$

Solution. The DE is separable.

$$\begin{aligned} \frac{dy}{dx} &= y^2 \cdot 1 \\ \int \frac{1}{y^2} dy &= \int 1 dx \\ -y^{-1} &= x + C \\ y &= -\frac{1}{x + C}. \end{aligned}$$

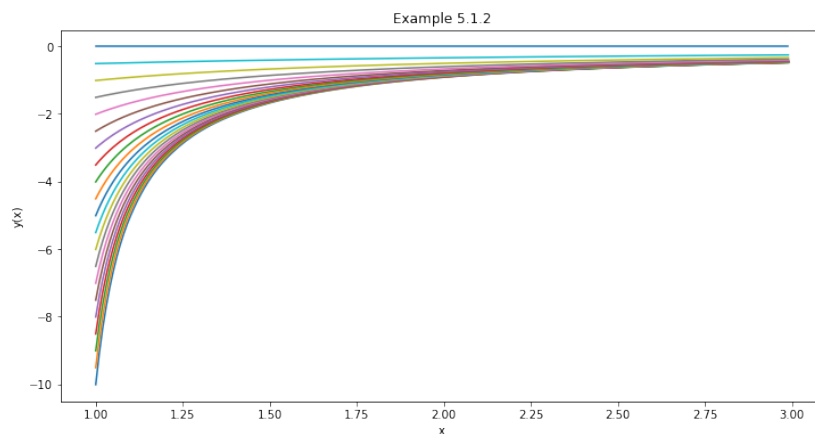
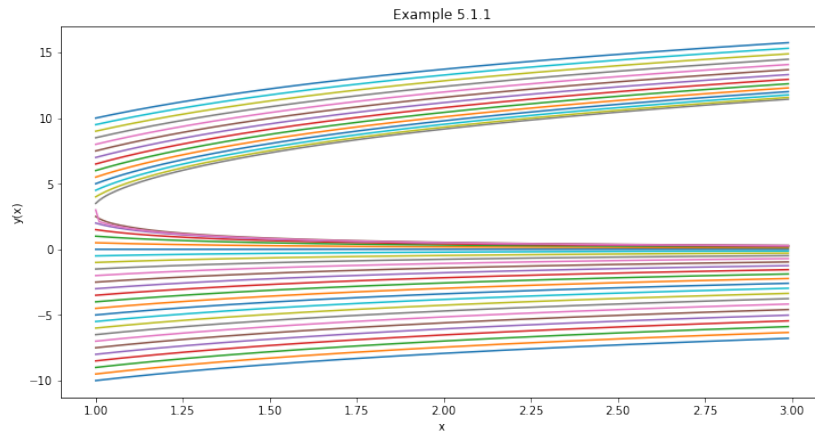
```
[2]: # Example 5.1
def f(x,y):
    return 4*y/(x*y-3*x)

def g(x,y):
    return y**2

# Solve the DEs numerically
sol1 = solve_ivp(f, [1, 3], np.arange(-10.01,10,0.5), t_eval=np.arange(1,3,0.01))
sol2 = solve_ivp(g, [1, 3], np.arange(-10.01,0,0.5), t_eval=np.arange(1,3,0.01))

# Plot the solutions
plt.figure(figsize = (12, 6))
for y in sol1.y:
    plt.plot(sol1.t,y)
plt.xlabel('x')
plt.ylabel('y(x)')
plt.title('Example 5.1.1')

plt.figure(figsize = (12, 6))
for y in sol2.y:
    plt.plot(sol2.t,y)
plt.xlabel('x')
plt.ylabel('y(x)')
plt.title('Example 5.1.2')
plt.show()
```



Example 5.2 Find the particular solution of the initial value problem.

$$\frac{dy}{dx} = \frac{x}{y^2}, \quad y(0) = 1.$$

Solution I. The DE is separable.

Solution II. The DE is separable.

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y^2} \\ \int y^2 dy &= \int x dx \\ \frac{1}{3}y^3 &= \frac{1}{2}x^2 + C \\ y &= \left(\frac{3}{2}x^2 + C\right)^{\frac{1}{3}}. \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y^2} \\ \int_1^y y^2 dy &= \int_0^x x dx \\ \frac{1}{3}y^3 - \frac{1}{3} &= \frac{1}{2}x^2 - 0 \\ y &= \left(\frac{3}{2}x^2 + 1\right)^{\frac{1}{3}}. \end{aligned}$$

Thus, $y(x) = \left(\frac{3}{2}x^2 + C\right)^{\frac{1}{3}}$ is the general solution. Since $y(0) = 1$, $C = 1$. Thus, the particular solution is

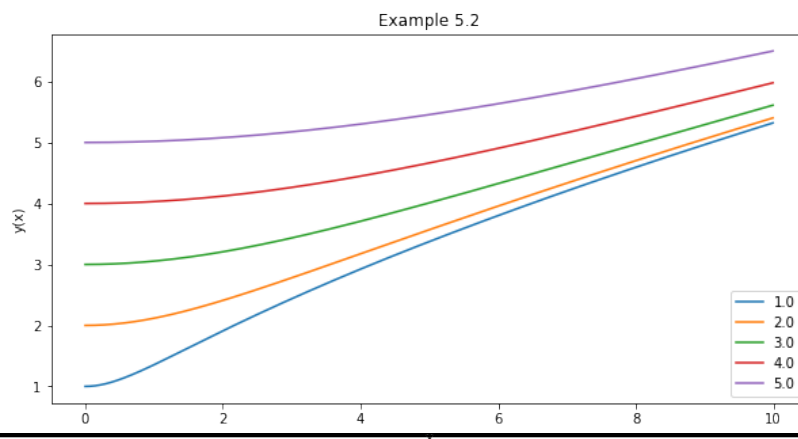
$$y(x) = \left(\frac{3}{2}x^2 + 1\right)^{\frac{1}{3}}.$$

Thus, $y(x) = \left(\frac{3}{2}x^2 + 1\right)^{\frac{1}{3}}$ is the particular solution of the DE.

```
[3]: # Example 5.2
def f(x,y):
    return x/y**2

# Solve the ODE numerically
sol = solve_ivp(f, [0, 10], [1,2,3,4,5], t_eval=np.arange(0,10,0.01))

# Plot the solutions
plt.figure(figsize = (10, 5))
for y in sol.y:
    plt.plot(sol.t,y,label=str(y[0]))
plt.xlabel('x')
plt.ylabel('y(x)')
plt.title('Example 5.2')
plt.legend(loc='best')
plt.show()
```



```
[1]: from sympy import *
```

```
[2]: # the variable x and the function y
x = symbols('x')
y = Function('y')(x)
```

```
[3]: # Example 5.1.1
DE = Eq(y.diff(), 4*y/(x*y-3*x)); DE
```

$$[3]: \frac{d}{dx} y(x) = \frac{4y(x)}{xy(x) - 3x}$$

```
[4]: # Solve the DE
dsolve(DE,y,simplify = False)
```

$$[4]: y(x) - 3 \log(y(x)) = C_1 + 4 \log(x)$$

```
[5]: # Default simplify = True
dsolve(DE,y)
```

```
[5]: [Eq(y(x), -3*LambertW(-(C1/x**4)**(1/3)/3)),
Eq(y(x), -3*LambertW((C1/x**4)**(1/3)*(1 -
↳sqrt(3)*I)/6)),
Eq(y(x), -3*LambertW((C1/x**4)**(1/3)*(1 +
↳sqrt(3)*I)/6))]
```

```
[6]: # Example 5.1.2
DE = Eq(y.diff(),y**2);DE
```

$$[6]: \frac{d}{dx} y(x) = y^2(x)$$

```
[7]: dsolve(DE,y,simplify = False)
```

$$[7]: -\frac{1}{y(x)} = C_1 + x$$

```
[8]: # Example 5.2
DE = Eq(y.diff(),x/y**2); DE
```

$$[8]: \frac{d}{dx} y(x) = \frac{x}{y^2(x)}$$

```
[9]: dsolve(DE,y,simplify = False)
```

$$[9]: \frac{y^3(x)}{3} = C_1 + \frac{x^2}{2}$$

```
[10]: # Solve the IVP
y = Function('y')
dsolve(DE,y(x),ics = {y(0):1})
```

$$[10]: y(x) = \sqrt[3]{\frac{3x^2}{2} + 1}$$

Example 5.3 Find the general solution of the DE

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

Solution. The DE is homogeneous since

$$\frac{dy}{dx} = \frac{1 - \left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

Let $y = ux$. Substituting these to DE yields

$$\begin{aligned} \int \frac{du}{\frac{1-u^2}{2u} - u} &= \int \frac{1}{x} dx \\ \int \frac{2u}{1-3u^2} du &= \int \frac{1}{x} dx \\ -\frac{1}{3} \ln|1-3u^2| &= \ln|x| + C \\ -\frac{1}{3} \ln\left|1 - \frac{3y^2}{x^2}\right| &= \ln|x| + C \\ y^2 &= \frac{x^3 + C}{3x} \end{aligned}$$

```
[11]: # Example 5.3
y = Function('y')(x)
DE = Eq(y.diff(),(x**2-y**2)/(2*x*y)); DE
```

```
[11]: d/dx y(x) = (x^2 - y^2(x)) / (2xy(x))
```

```
[12]: dsolve(DE,y,simplify = False)
```

```
[12]: y^2(x) = (C1 + x^3) / x
```

Example 5.4 Find the general solution of the DE

$$\frac{dy}{dx} = \frac{-2y}{3x - y}$$

Solution. The DE is homogeneous since

$$\frac{dy}{dx} = \frac{-2\left(\frac{y}{x}\right)}{3 - \left(\frac{y}{x}\right)}$$

Let $y = ux$. Substituting these to DE yields

$$\begin{aligned} \int \frac{du}{\frac{-2u}{3-u} - u} &= \int \frac{1}{x} dx \\ \int \frac{3-u}{u(u-5)} du &= \int \frac{1}{x} dx \\ -\frac{1}{5} (3 \ln|u| + 2 \ln|u-5|) &= \ln|x| + C \\ -\frac{1}{5} \left(3 \ln\left|\frac{y}{x}\right| + 2 \ln\left|\frac{y}{x} - 5\right|\right) &= \ln|x| + C \\ 3 \ln|y| + 2 \ln|y - 5x| &= C \end{aligned}$$

```
[13]: # Example 5.4
DE = Eq(y.diff(),-2*y/(3*x-y)); DE
```

```
[13]: d/dx y(x) = -2y(x) / (3x - y(x))
```

```
[14]: dsolve(DE,y,simplify = False)
```

```
[14]: log(y(x)) = log(C1) - 2*log(5*x/y(x) - 1) / 5
```

Example 5.5 Solve the following linear equation

$$\frac{dy}{dx} + \frac{2}{x}y = x - 1 \quad y(1) = 0.$$

Solution. The DE is linear and the **integrating factor** is

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2.$$

Now the general solution is

$$y = \frac{1}{x^2} \int x^2(x-1) dx$$

$$y = \frac{1}{x^2} \left(\frac{x^4}{4} - \frac{x^3}{3} + C \right)$$

$$y = \frac{x^2}{4} - \frac{x}{3} + \frac{C}{x^2}.$$

```
[15]: # Example 5.5
DE = Eq(y.diff()+2*y/x,x-1); DE
```

```
[15]: d/dx y(x) + 2y(x)/x = x - 1
```

```
[16]: dsolve(DE,y,simplify = False)
```

```
[16]: y(x) = C1/x^2 + x^2/4 - x/3
```

```
[17]: y = Function('y')
dsolve(DE,y(x),simplify = False, ics={y(1):0})
```

```
[17]: y(x) = x^2/4 - x/3 + 1/(12x^2)
```

Since $y(1) = 0$, $0 = \frac{1}{4} - \frac{1}{3} + C$ and hence $C = \frac{1}{12}$, i.e., $y = \frac{x^2}{4} - \frac{x}{3} + \frac{1}{12x^2}$.

Example 5.6 Solve the following linear equation

$$(1+x^2)y' + (1-x)^2y = xe^{-x}.$$

Solution. The DE is linear since it can be rewritten as

$$y' + \frac{(1-x)^2}{1+x^2}y = \frac{xe^{-x}}{1+x^2}.$$

Then the integrating factor is

$$\mu(x) = e^{\int \frac{(1-x)^2}{1+x^2} dx} = e^{x - \ln|1+x^2|} = e^x e^{\ln(1+x^2)^{-1}} = \frac{e^x}{1+x^2}.$$

Now the general solution is

$$\begin{aligned} y &= \frac{1+x^2}{e^x} \int \frac{e^x}{1+x^2} \cdot \frac{xe^{-x}}{1+x^2} dx = \frac{1+x^2}{e^x} \int \frac{x}{(1+x^2)^2} dx \\ &= \frac{1+x^2}{e^x} \left(-\frac{1}{2(1+x^2)} + C \right) = e^{-x} \left(C(1+x^2) - \frac{1}{2} \right). \end{aligned}$$

```
[18]: # Example 5.6
y = Function('y')(x)
DE = Eq((1+x**2)*y.
->diff()+(1-x)**2*y,x*exp(-x)); DE
```

```
[18]: (1 - x)^2 y(x) + (x^2 + 1) d/dx y(x) = xe^{-x}
```

```
[19]: dsolve(DE,y,simplify = True)
```

```
[19]: y(x) = (C1x^2 + C1 - 1/2) e^{-x}
```

Example 5.7 Solve the following DE.

$$(y^2 - x^2) + (2xy + 1) \frac{dy}{dx} = 0.$$

[20]: `DE = Eq(y**2-x**2+(2*x*y+1)*y.diff(),0); DE`

[20]: $-x^2 + (2xy(x) + 1) \frac{d}{dx}y(x) + y^2(x) = 0$

[21]: `dsolve(DE,y,simplify = False)`

[21]: $-\frac{x^3}{3} + xy^2(x) + y(x) = C_1$

Solution. Let

$$M(x, y) = y^2 - x^2 \quad \text{and} \quad N(x, y) = 2xy + 1.$$

Then

$$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y$$

and hence the DE is exact. Then

$$\begin{aligned} f(x, y) &= \int y^2 - x^2 dx + \int (2xy + 1 - \int 2y dx) dy \\ &= xy^2 - \frac{1}{3}x^3 + \int 2xy + 1 - 2xy dy = xy^2 - \frac{1}{3}x^3 + y. \end{aligned}$$

Then the general solution of DE is

$$f(x, y) = xy^2 - \frac{1}{3}x^3 + y = C.$$

Example 5.8 Solve the following DE.

$$\frac{dy}{dx} = -\frac{2y - 1}{2x - y + 2}.$$

[22]: `# Example 5.8`
`DE = Eq(y.diff(),-(2*y-1)/(2*x-y+2)); DE`

[22]: $\frac{d}{dx}y(x) = \frac{1 - 2y(x)}{2x - y(x) + 2}$

[23]: `dsolve(DE,y,simplify = False)`

Solution. Rewrite the DE

as

$$(2y-1) + (2x-y+2) \frac{dy}{dx} = 0.$$

[23]: $\log\left(y(x) - \frac{1}{2}\right) = \log(C_1) - \frac{\log\left(\frac{4\left(x + \frac{3}{4}\right)}{y(x) - \frac{1}{2}} - 1\right)}{2}$

Let

$$M(x, y) = 2y - 1 \quad \text{and} \quad N(x, y) = 2x - y + 2 \implies M_y = 2 \quad \text{and} \quad N_x = 2$$

and hence the DE is exact. Then

$$\begin{aligned} y &= \int 2y - 1 dx + \int (2x - y + 2 - \int 2 dx) dy \\ &= 2xy - x + \int 2x - y + 2 - 2x dy = 2xy - x - \frac{1}{2}y^2 + 2y. \end{aligned}$$

Then the general solution of DE is

$$f(x, y) = 2xy - x - \frac{1}{2}y^2 + 2y = C.$$

Example 5.8 [Re-visited] Solve the following DE.

$$\frac{dy}{dx} = -\frac{2y - 1}{2x - y + 2}$$

Solution. Let $u = x + \frac{3}{4}$ and $v = y - \frac{1}{2}$. Then $\frac{dy}{dx} = \frac{dv}{du}$ and the DE becomes

$$\frac{dv}{du} = -\frac{2v}{2u - v} = \frac{2\left(\frac{v}{u}\right)}{\left(\frac{v}{u}\right) - 2},$$

which is homogeneous. Let $v = wu$. Then

$$\int \frac{dw}{\frac{2w}{w-2} - w} = \int \frac{1}{u} du$$

$$\int \frac{w-2}{w(4-w)} dw = \int \frac{1}{u} du$$

$$-\frac{1}{2} \ln |w(w-4)| = \ln |u| + C$$

$$\ln |uw| + \ln |uw - 4u| = C$$

$$\ln \left| y - \frac{1}{2} \right| + \ln \left| y - 4x - \frac{7}{2} \right| = C$$

$$y^2 - 4xy - 4y + 2x = C.$$

Example Solve the following DE.

$$(2x^2y + y^2) + (2x^3 - xy) \frac{dy}{dx} = 0.$$

[24]: `DE = Eq(2*x**2*y+y**2+(2*x**3-x*y)*y.diff(),0); DE`

[24]: $2x^2y(x) + (2x^3 - xy(x)) \frac{d}{dx} y(x) + y^2(x) = 0$

[25]: `dsolve(DE,y,simplify = False)`

[25]: `[Eq(y(x), (32*2**(1/3)*x**4 + ((-x**3 + (16*x**6 + sqrt(x**6*(-256*x**6*exp(6*C1) + (16*x**3*exp(3*C1) - 1)**2)*exp(-6*C1)))*exp(3*C1)*exp(-3*C1))**(1/3)*(1 - sqrt(3)*I)*(16*x**2 + 2**(2/3)*((-x**3 + (16*x**6 + sqrt(x**6*(-256*x**6*exp(6*C1) + (16*x**3*exp(3*C1) - 1)**2)*exp(-6*C1)))*exp(3*C1)*exp(-3*C1))**(1/3)*(1 - sqrt(3)*I)))/(4*((-x**3 + (16*x**6 + sqrt(x**6*(-256*x**6*exp(6*C1) + (16*x**3*exp(3*C1) - 1)**2)*exp(-6*C1)))*exp(3*C1)*exp(-3*C1))**(1/3)*(1 - sqrt(3)*I))), Eq(y(x), (32*2**(1/3)*x**4 + ((-x**3 + (16*x**6 + sqrt(x**6*(-256*x**6*exp(6*C1) + (16*x**3*exp(3*C1) - 1)**2)*exp(-6*C1)))*exp(3*C1)*exp(-3*C1))**(1/3)*(1 + sqrt(3)*I)*(16*x**2 + 2**(2/3)*((-x**3 + (16*x**6 + sqrt(x**6*(-256*x**6*exp(6*C1) + (16*x**3*exp(3*C1) - 1)**2)*exp(-6*C1)))*exp(3*C1)*exp(-3*C1))**(1/3)*(1 + sqrt(3)*I)))/(4*((-x**3 + (16*x**6 + sqrt(x**6*(-256*x**6*exp(6*C1) + (16*x**3*exp(3*C1) - 1)**2)*exp(-6*C1)))*exp(3*C1)*exp(-3*C1))**(1/3)*(1 + sqrt(3)*I))), Eq(y(x), -4*2**(1/3)*x**4/(16*x**6 - x**3*exp(-3*C1) + sqrt(-32*x**9*exp(-3*C1) + x**6*exp(-6*C1))**(1/3) + 4*x**2 - 2**(2/3)*(16*x**6 - x**3*exp(-3*C1) + sqrt(-32*x**9*exp(-3*C1) + x**6*exp(-6*C1))**(1/3)/2)]`

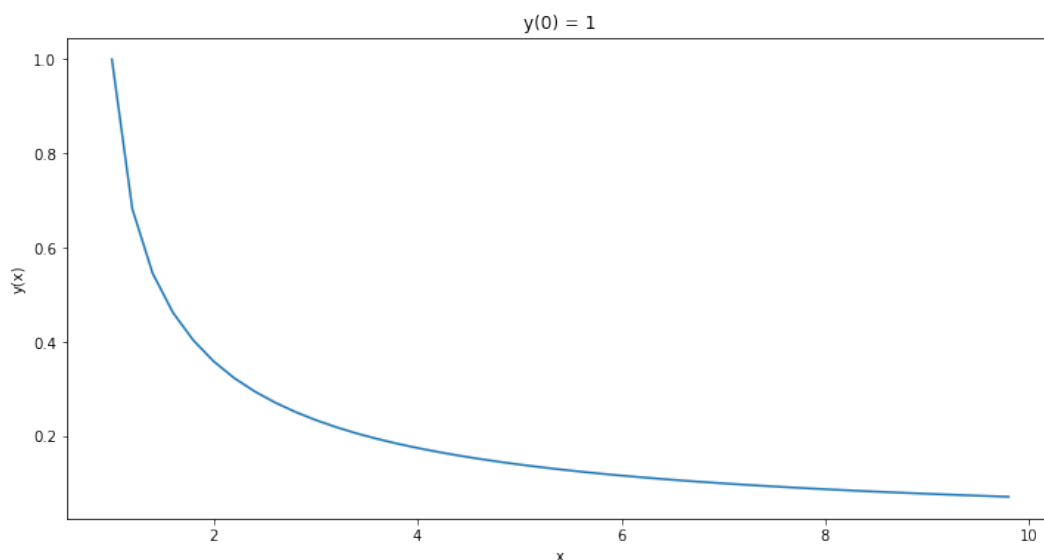
```
[26]: import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

# define the function f(x,y)
def f(x,y):
    return (2*x**2*y+y**2)/(x*y-2*x**3)

# Solve the ODE numerically with y(0) = 1
sol = solve_ivp(f, [1, 10], [1], t_eval=np.arange(1,10,.2));
sol
```

```
[26]: message: 'The solver successfully reached the end of the integration
interval.'
      nfev: 44
      njev: 0
      nlu: 0
      sol: None
      status: 0
      success: True
      t: array([1. , 1.2, 1.4, 1.6, 1.8, 2. , 2.2, 2.4, 2.6, 2.8, 3. , 3.2, 3.4,
3.6, 3.8, 4. , 4.2, 4.4, 4.6, 4.8, 5. , 5.2, 5.4, 5.6, 5.8, 6. ,
6.2, 6.4, 6.6, 6.8, 7. , 7.2, 7.4, 7.6, 7.8, 8. , 8.2, 8.4, 8.6,
8.8, 9. , 9.2, 9.4, 9.6, 9.8])
      t_events: None
      y: array([[1.          , 0.68252086, 0.54530629, 0.4617929 , 0.40303162,
0.3583015 , 0.32316716, 0.29484305, 0.27128892, 0.25120092,
0.23394037, 0.21891177, 0.20573463, 0.19411553, 0.18379757,
0.17456034, 0.16621992, 0.15862892, 0.15167641, 0.14528801,
0.13942546, 0.13402421, 0.12901704, 0.1243696 , 0.12004967,
0.11602713, 0.11227397, 0.1087643 , 0.10547434, 0.10238245,
0.09946907, 0.09671678, 0.09411026, 0.0916363 , 0.08928384,
0.08704389, 0.08490959, 0.08287622, 0.08094114, 0.07909945,
0.07733994, 0.07565711, 0.07404605, 0.07250222, 0.07102145]])
      y_events: None
```

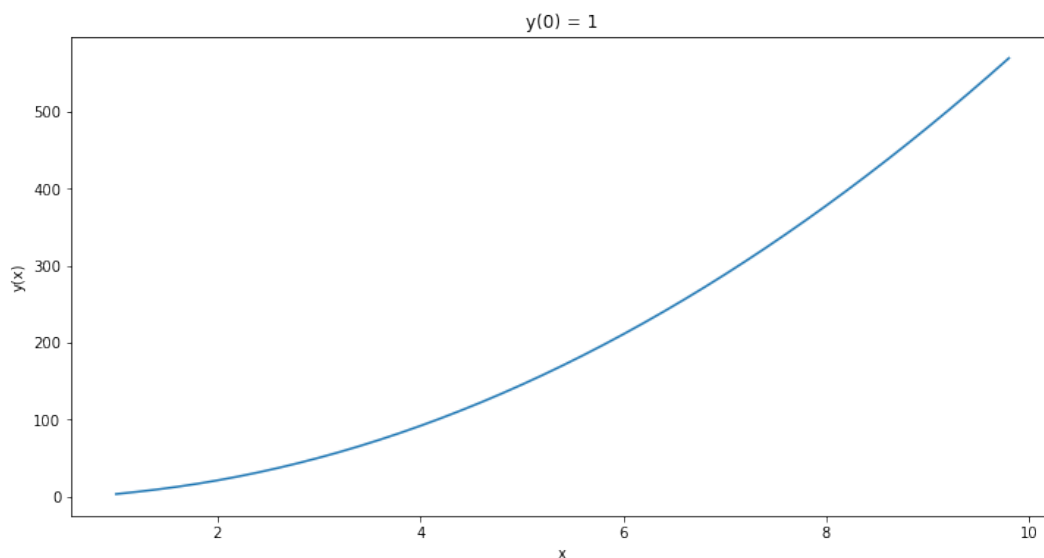
```
[27]: # Plot the solution
plt.figure(figsize = (12, 6))
for y in sol.y:
    plt.plot(sol.t,y)
plt.xlabel('x')
plt.ylabel('y(x)')
plt.title('y(0) = 1')
plt.show()
```



```
[28]: # Solve the ODE numerically with  $y(0) = 3$ 
sol = solve_ivp(f, [1, 10], [3], t_eval=np.arange(1,10,.2));
sol
```

```
[28]: message: 'The solver successfully reached the end of the integration
interval.'
      nfev: 38
      njev: 0
      nlu: 0
      sol: None
      status: 0
      success: True
      t: array([1. , 1.2, 1.4, 1.6, 1.8, 2. , 2.2, 2.4, 2.6, 2.8, 3. , 3.2,
3.4,
3.6, 3.8, 4. , 4.2, 4.4, 4.6, 4.8, 5. , 5.2, 5.4, 5.6, 5.8, 6. ,
6.2, 6.4, 6.6, 6.8, 7. , 7.2, 7.4, 7.6, 7.8, 8. , 8.2, 8.4, 8.6,
8.8, 9. , 9.2, 9.4, 9.6, 9.8])
      t_events: None
      y: array([[ 3.          ,  5.80484654,  8.87037149, 12.36982742,
16.33549591, 20.77673072, 25.69652807, 31.09657108,
36.97810945, 43.34195954, 50.18850428, 57.51771353,
65.33025834, 73.62609908, 82.40458002, 91.66512464,
101.4072356 , 111.63049478, 122.33456323, 133.51918123,
145.18416823, 157.32942288, 169.95492305, 183.06072579,
196.64696734, 210.71386315, 225.26170787, 240.29087533,
255.80181859, 271.79506987, 288.27124061, 305.23101602,
322.67317191, 340.5963153 , 359.00040087, 377.88538732,
397.25123741, 417.09791794, 437.42539974, 458.23365769,
479.52267071, 501.29242177, 523.54289787, 546.27409005,
569.48599341]])
      y_events: None
```

```
[29]: # Plot the solution
plt.figure(figsize = (12, 6))
for y in sol.y:
    plt.plot(sol.t,y)
plt.xlabel('x')
plt.ylabel('y(x)')
plt.title('y(0) = 3')
plt.show()
```



Example 5.9 (a) Show that $y(x) = C \exp\left(-\int a(x) dx\right)$ is a solution of the linear DE

$$y' + a(x)y = 0.$$

(b) Assume that $y(x) = C(x) \exp\left(-\int a(x) dx\right)$ is a solution of the linear DE

$$y' + a(x)y = b(x).$$

Show that $C(x)$ satisfies that condition

$$C'(x) = b(x) \exp\left(\int a(x) dx\right).$$

Solution. (a)

$$\begin{aligned} y' + a(x)y &= 0 & y &= \exp\left(-\int a(x) dx + C\right) \\ \frac{dy}{dx} &= -a(x)y & y &= C \exp\left(-\int a(x) dx\right). \\ \int \frac{1}{y} dy &= -\int a(x) dx \\ \ln |y| &= -\int a(x) dx + C \end{aligned}$$

(b) Assume that $y(x) = C(x) \exp\left(-\int a(x) dx\right)$ is a solution. Then

$$\begin{aligned} y'(x) &= \frac{d}{dx} \left(C(x) \exp\left(-\int a(x) dx\right) \right) \\ &= C'(x) \exp\left(-\int a(x) dx\right) - a(x) C(x) \exp\left(-\int a(x) dx\right) \\ &= C'(x) \exp\left(-\int a(x) dx\right) - a(x)y(x) \\ y'(x) + a(x)y(x) &= C'(x) \exp\left(-\int a(x) dx\right). \end{aligned}$$

Therefore,

$$C'(x) \exp\left(-\int a(x) dx\right) = b(x) \implies C'(x) = b(x) \exp\left(\int a(x) dx\right).$$

Existence Theorem

Suppose the two variable function $f(x, y)$ is a **continuous** function in a rectangle $[a, b] \times [c, d]$. If (x_0, y_0) is a point in this rectangle, then there exists $\epsilon > 0$ and a function $y(x)$ defined in $(x_0 - \epsilon, x_0 + \epsilon)$ that satisfies initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.$$

Uniqueness Theorem

Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}$ are **continuous** function in a rectangle $[a, b] \times [c, d]$. If (x_0, y_0) is a point in this rectangle and if both $y_1(x)$ and $y_2(x)$ are TWO solutions of the initial-value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

for all x in $(x_0 - \epsilon, x_0 + \epsilon)$, then

$$y_1(x) = y_2(x) \quad \text{for all } x \in (x_0 - \epsilon, x_0 + \epsilon).$$

That is, the solution is **unique**.

Example Consider the following initial-value problem

$$\frac{dy}{dx} = 3y^{\frac{2}{3}}, \quad y(0) = 0.$$

- ▶ Clearly, $y(x) = 0$ is a solution of the above IVP since

$$\text{L.H.S.} = \frac{dy}{dx} = 0 \quad \text{and} \quad \text{R.H.S.} = 3y^{\frac{2}{3}} = 0.$$

- ▶ On the other hand,

$$\begin{aligned} \frac{dy}{dx} &= 3y^{\frac{2}{3}} \\ \int_0^y y^{-\frac{2}{3}} dy &= \int_0^x 3 dx \\ 3y^{\frac{1}{3}} &= 3x \\ y &= x^3 \end{aligned}$$

[24]: `DE = Eq(3*y**S('2/3'),0); DE`

[24]: `3y2/3(x) = 0`

[25]: `dsolve(DE,y,simplify = False)`

[25]: `y(x) = 0`

So $y(x) = x^3$ is another solution!!

The Uniqueness Theorem **fails** because

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left(3y^{\frac{2}{3}} \right) = 2y^{-\frac{1}{3}},$$

which is NOT continuous at $y = 0$.

Second order ordinary differential equation

- ▶ A **second order linear differential equation** is an equation of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x),$$

where $P(x)$, $Q(x)$, and $R(x)$ are functions of x .

- ▶ When $P(x)$ and $Q(x)$ are constant functions, i.e.,

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = R(x)$$

for some constants p and q , this equation is called a **second order linear differential equation with constant coefficients**.

- ▶ When $R(x) = 0$, the equation is called the second order linear **homogeneous** differential equation. Otherwise, it is **non-homogeneous**.

Existence and Uniqueness Theorem

Consider the initial value problem

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x), \quad y(x_0) = y_0 \text{ and } y'(x_0) = y'_0,$$

where $P(x)$, $Q(x)$, and $R(x)$ are **continuous** on an open interval containing x_0 . Then there exists an **unique** solution $y(x)$ defined in the interval satisfying the above problem.

Example Given the second order initial value problem

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-x}, \quad y(0) = 0 \text{ and } y'(0) = 0.$$

The general solution of the DE is

$$y(x) = C_1e^{-3x} + C_2e^{-2x} + \frac{1}{2}e^{-x},$$

and the particular solution of the initial value problem is

$$y(x) = \frac{1}{2}e^{-3x} - e^{-2x} + \frac{1}{2}e^{-x}.$$

Second order ordinary differential equation

P. 37 / 77

```
[1]: from sympy import *  
x = symbols('x')  
y = Function('y')(x)
```

```
[2]: DE = Eq(y.diff(x,x)+5*y.diff(x)+6*y,exp(-x));DE
```

$$[2]: 6y(x) + 5\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = e^{-x}$$

```
[3]: dsolve(DE,y,simplify = False)
```

$$[3]: y(x) = C_1e^{-3x} + C_2e^{-2x} + \frac{e^{-x}}{2}$$

```
[4]: y = Function('y')  
dsolve(DE,y(x),ics = {y(0):0,y(x).diff(x).subs(x,0):0},simplify = False)
```

$$[4]: y(x) = \frac{e^{-x}}{2} - e^{-2x} + \frac{e^{-3x}}{2}$$

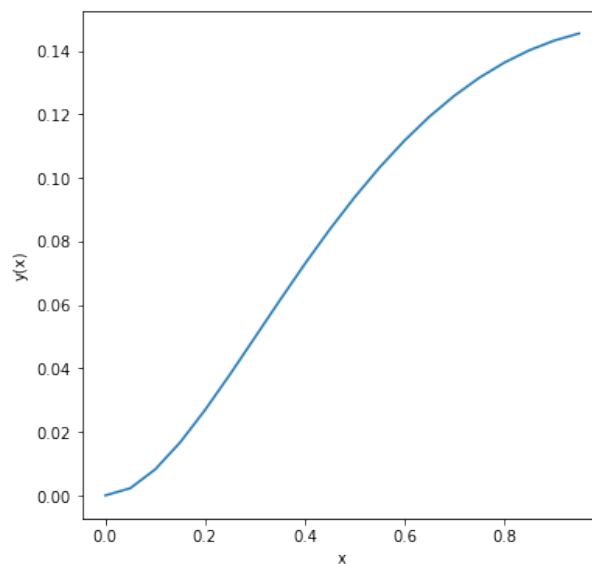
```
[5]: # define the function f(x,Y) with Y = [y,z] and z = dy/dx  
import numpy as np  
from scipy.integrate import solve_ivp  
  
def f(x,Y):  
    y = Y[0]  
    z = Y[1]  
    dy = z  
    dz = -6*y-5*z+2*np.exp(-x)  
    dY = np.array([dy, dz])  
    return dY
```

Second order ordinary differential equation

P. 38 / 77

```
[6]: # Solve the ODE numerically  
sol = solve_ivp(f, [0, 1], np.array([0,0]), t_eval=np.arange(0,1,0.05))
```

```
[7]: # Plot the solution  
plt.figure(figsize = (6, 6))  
plt.plot(sol.t,sol.y[0])  
plt.xlabel('x')  
plt.ylabel('y(x)')  
plt.show()
```



Principle of Superposition

Suppose $y_1(x)$ and $y_2(x)$ are two solutions of the linear homogeneous differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

Let

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Then $y(x)$ is also a solution of the linear homogeneous differential equation.

Proof. Notice that

$$\begin{aligned} & \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y \\ = & \frac{d^2}{dx^2}(c_1 y_1 + c_2 y_2) + P(x)\frac{d}{dx}(c_1 y_1 + c_2 y_2) + Q(x)(c_1 y_1 + c_2 y_2) \\ = & c_1 \left(\frac{d^2 y_1}{dx^2} + P(x)\frac{d y_1}{dx} + Q(x)y_1 \right) + c_2 \left(\frac{d^2 y_2}{dx^2} + P(x)\frac{d y_2}{dx} + Q(x)y_2 \right) \\ = & c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

Therefore, $y(x)$ is also a solution of the same equation.

The general solution of homogeneous equation

Suppose $y_1(x)$ and $y_2(x)$ are two solutions of the linear differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

If the **Wronskian** of y_1 and y_2 ,

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x),$$

is **nonzero** for some $x = x_0$, i.e., $W(y_1, y_2)(x_0) \neq 0$, then $y_1(x)$ and $y_2(x)$ are **two linearly independent** solutions. Furthermore, every solution $y(x)$ of the above equation can be expressed as

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Proof. Suppose $\hat{y}(x)$ is a solution of the above linear homogeneous DE. Let $y_0 = \hat{y}(x_0)$ and $y'_0 = \hat{y}'(x_0)$ and consider the initial value problem

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y(x_0) = y_0 \text{ and } y'(x_0) = y'_0.$$

Clearly, $\hat{y}(x)$ is a solution of this initial value problem on an open interval containing x_0 . Now consider the system of linear equations and by Cramer's rule

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = y_0' \end{cases} \implies \begin{cases} c_1 = \frac{\begin{vmatrix} y_0 & y_2(x_0) \\ y_0' & y_2'(x_0) \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}} \\ c_2 = \frac{\begin{vmatrix} y_1(x_0) & y_0 \\ y_1'(x_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}} \end{cases}$$

Therefore, for such c_1 and c_2 , $y(x) = c_1 y_1(x) + c_2 y_2(x)$ also satisfies the initial conditions of the above initial value problem. By the uniqueness theorem,

$$\hat{y}(x) = c_1 y_1(x) + c_2 y_2(x)$$

on an open interval containing x_0 .

Example 5.10 Show that $y_1(x) = x^{\frac{1}{2}}$ and $y_2(x) = x^{-1}$ are two linearly independent solutions of

$$2x^2 y'' + 3xy' - y = 0, \quad x > 0.$$

Solution. Direct substitutions show that

$$2x^2 y_1'' + 3xy_1' - y_1 = 2x^2 \left(-\frac{1}{4}x^{-\frac{3}{2}} \right) + 3x \left(\frac{1}{2}x^{-\frac{1}{2}} \right) - x^{\frac{1}{2}} = 0$$

$$2x^2 y_2'' + 3xy_2' - y_2 = 2x^2 (2x^{-3}) + 3x (-x^2) - x^{-1} = 0.$$

Therefore, y_1 and y_2 are two solutions of the DE. Now since $x > 0$,

$$W(y_1, y_2)(x) = \begin{vmatrix} x^{\frac{1}{2}} & x^{-1} \\ \frac{1}{2}x^{-\frac{1}{2}} & -x^{-2} \end{vmatrix} = -x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = -\frac{3}{2}x^{-\frac{3}{2}} \neq 0.$$

Thus, y_1 and y_2 are two linearly independent solutions of the DE.

Second order linear homogeneous differential equation P. 43 / 77

Given the second order linear homogeneous differential equation with constant coefficients

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0.$$

Let a and b be the roots of the characteristic equation

$$\lambda^2 + p\lambda + q = 0.$$

1. If a and b are real and distinct, then $y_1(x) = e^{ax}$ and $y_2(x) = e^{bx}$ are two linearly independent solutions and the general solution is

$$y(x) = c_1e^{ax} + c_2e^{bx}.$$

2. If a and b are real and equal, i.e., $a = b$, then $y_1(x) = e^{ax}$ and $y_2(x) = xe^{ax}$ are two linearly independent solutions and the general solution is

$$y(x) = c_1e^{ax} + c_2xe^{ax} = (c_1 + c_2x)e^{ax}.$$

3. If $a = \alpha + i\beta$ and $b = \alpha - i\beta$ are complex roots, then $y_1(x) = e^{\alpha x} \cos \beta x$ and $y_2(x) = e^{\alpha x} \sin \beta x$ are two linearly independent solutions and the general solution is

$$y(x) = c_1e^{\alpha x} \cos \beta x + c_2e^{\alpha x} \sin \beta x = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x).$$

Second order linear homogeneous differential equation P. 44 / 77

Given a homogeneous equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0.$$

Let

$$y = e^{\lambda x}, \quad \frac{dy}{dx} = \lambda e^{\lambda x}, \quad \text{and} \quad \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}.$$

Then

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x} = e^{\lambda x} (\lambda^2 + p\lambda + q).$$

It follows that $y = e^{\lambda x}$ is a solution of the DE if λ satisfies the quadratic equation

$$\lambda^2 + p\lambda + q = 0,$$

which is called the **characteristic equation**.

1. If a and b are **real and distinct**, since

$$W(e^{ax}, e^{bx}) = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = (b-a)e^{(a+b)x} \neq 0,$$

$y_1(x) = e^{ax}$ and $y_2(x) = e^{bx}$ are two **linearly independent** solutions of the homogeneous DE. Then the general solution is

$$y(x) = c_1e^{ax} + c_2e^{bx}.$$

2. If a and b are **real and equal**, i.e., $a = b$, then

$$y_1(x) = e^{ax} \text{ is the first solution.}$$

Consider $y_2(x) = xe^{ax}$. Then

$$\frac{dy_2}{dx} = e^{ax} + axe^{ax} \quad \text{and} \quad \frac{d^2y_2}{dx^2} = 2ae^{ax} + a^2xe^{ax}.$$

and so

$$\begin{aligned} \frac{d^2y_2}{dx^2} + p\frac{dy_2}{dx} + qy_2 &= 2ae^{ax} + a^2xe^{ax} + p(e^{ax} + axe^{ax}) + qxe^{ax} \\ &= (2a + p)e^{ax} + (a^2 + pa + q)xe^{ax} = 0. \end{aligned}$$

So $y_2(x) = xe^{ax}$ is another solution. Now

$$W(e^{ax}, xe^{ax}) = \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & (ax + 1)e^{ax} \end{vmatrix} = e^{2ax} \neq 0.$$

Therefore, $y_1(x) = e^{ax}$ and $y_2(x) = xe^{ax}$ are two **linearly independent** solutions and the general solution is

$$y(x) = c_1e^{ax} + c_2xe^{ax}.$$

3. If $a = \alpha + i\beta$ and $b = \alpha - i\beta$ are **two complex roots**, the two solutions are

$$y_1(x) = e^{(\alpha+i\beta)x} \quad \text{and} \quad y_2(x) = e^{(\alpha-i\beta)x}.$$

Notice that

$$e^{x+iy} = e^x(\cos y + i \sin y) \quad \text{and} \quad e^{x-iy} = e^x(\cos y - i \sin y).$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= k_1e^{\alpha x}(\cos \beta x + i \sin \beta x) + k_2e^{\alpha x}(\cos \beta x - i \sin \beta x) \\ &= (k_1 + k_2)e^{\alpha x} \cos \beta x + i(k_1 - k_2)e^{\alpha x} \sin \beta x. \end{aligned}$$

Now

$$\begin{aligned} W(e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x) &= \begin{vmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ e^{\alpha x}(\alpha \cos \beta x - \beta \sin \beta x) & e^{\alpha x}(\alpha \sin \beta x + \beta \cos \beta x) \end{vmatrix} \\ &= \beta e^{2\alpha x} \neq 0. \end{aligned}$$

Take $c_1 = k_1 + k_2$ and $c_2 = i(k_1 - k_2)$. Then $y_1(x) = e^{\alpha x} \cos \beta x$ and $y_2(x) = e^{\alpha x} \sin \beta x$ are two **linearly independent** solutions and the general solution is

$$y(x) = c_1e^{\alpha x} \cos \beta x + c_2e^{\alpha x} \sin \beta x = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x).$$

Example For the homogeneous DE

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0,$$

the characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0 \implies \lambda = -2 \text{ or } -3.$$

The general solution for the DE is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{-3x}.$$

Example For the homogeneous DE

$$y'' + y = 0,$$

the characteristic equation is

$$\lambda^2 + 1 = 0 \implies \lambda = \pm i.$$

The general solution for the DE is

$$y(x) = c_1 \cos x + c_2 \sin x.$$

```
[1]: DE = Eq(y.diff(x,x) + 5*y,
      ↪diff(x)+6*y,0); DE
```

```
[1]: 6y(x) + 5 d/dx y(x) + d^2/dx^2 y(x) = 0
```

```
[2]: dsolve(DE,y,simplify = False)
```

```
[2]: y(x) = C1 e^{-3x} + C2 e^{-2x}
```

```
[3]: DE = Eq(y.diff(x,x) + y,0); DE
```

```
[3]: y(x) + d^2/dx^2 y(x) = 0
```

```
[4]: dsolve(DE,y,simplify = False)
```

```
[4]: y(x) = C1 sin(x) + C2 cos(x)
```

General solution - Non-homogeneous case

Suppose $y_p(x)$ is a **particular solution** of the non-homogeneous equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x).$$

and $y_1(x)$ and $y_2(x)$ are **two linearly independent solutions** of the **associated homogeneous equation**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

then the **general solution** of the non-homogeneous equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Example Given the second order initial value problem

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-x}.$$

The general solution of the DE is

$$y(x) = c_1 \underbrace{e^{-3x}}_{y_1(x)} + c_2 \underbrace{e^{-2x}}_{y_2(x)} + \underbrace{\frac{1}{2}e^{-x}}_{y_p(x)}.$$

Proof. Given a non-homogeneous equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x).$$

Suppose $y_p(x)$ and $y_q(x)$ are two particular solutions of the non-homogeneous equation. Then

$$y_p'' + Py_p' + Qy_p = R \quad \text{and} \quad y_q'' + Py_q' + Qy_q = R.$$

Let $y_h(x) = y_q(x) - y_p(x)$. Then

$$\begin{aligned} y_h'' + Py_h' + Qy_h &= (y_q'' - y_p'') + P(y_q' - y_p') + Q(y_q - y_p) \\ &= (y_q'' + Py_q' + Qy_q) - (y_p'' + Py_p' + Qy_p) \\ &= R - R = 0. \end{aligned}$$

i.e., $y_h(x)$ is a solution of the associated homogeneous equation. Thus, $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$. Equivalently,

$$y_q(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Variation of parameters

Given a non-homogeneous DE

$$y'' + P(x)y' + Q(x)y = R(x).$$

Suppose $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the associated homogeneous equation. Let

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x).$$

Assume

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = R(x). \end{cases}$$

Then

$$\begin{aligned} y_p(x) &= v_1y_1 + v_2y_2, \\ y_p'(x) &= v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2' = v_1y_1' + v_2y_2' + 0 \\ y_p''(x) &= v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' = v_1y_1'' + v_2y_2'' + R(x). \end{aligned}$$

Therefore,

$$\begin{aligned} y_p'' + Py_p' + Qy_p &= (v_1y_1'' + v_2y_2'' + R(x)) + P(v_1y_1' + v_2y_2') + Q(v_1y_1 + v_2y_2) \\ &= v_1(y_1'' + Py_1' + Qy_1) + v_2(y_2'' + Py_2' + Qy_2) + R(x) \\ &= v_1 \cdot 0 + v_2 \cdot 0 + R(x) = R(x). \end{aligned}$$

It follows that $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$ is a particular solution of the non-homogeneous equation if

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ v_1'y_1' + v_2'y_2' = R(x) \end{cases} \implies \begin{cases} y_1(v_1') + y_2(v_2') = 0 \\ y_1'(v_1') + y_2'(v_2') = R(x). \end{cases}$$

Solving the above simultaneously equations, we obtain

$$v_1'(x) = -\frac{y_2(x)R(x)}{W(y_1, y_2)} \quad \text{and} \quad v_2'(x) = \frac{y_1(x)R(x)}{W(y_1, y_2)},$$

where $W(y_1, y_2) = y_1 y_2' - y_1' y_2$. Then

$$v_1(x) = -\int \frac{y_2(x)R(x)}{W(y_1, y_2)} dx \quad \text{and} \quad v_2(x) = \int \frac{y_1(x)R(x)}{W(y_1, y_2)} dx.$$

Therefore, a particular solution of the non-homogeneous equation is

$$y_p(x) = \left(-\int \frac{y_2(x)R(x)}{W(y_1, y_2)} dx \right) y_1(x) + \left(\int \frac{y_1(x)R(x)}{W(y_1, y_2)} dx \right) y_2(x).$$

In summary, the general solution of the non-homogeneous equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \left(-\int \frac{y_2(x)R(x)}{W(y_1, y_2)} dx \right) y_1(x) + \left(\int \frac{y_1(x)R(x)}{W(y_1, y_2)} dx \right) y_2(x).$$

In particular, for second order linear non-homogeneous DE with constant coefficients

$$y'' + py' + qy = R(x),$$

- ▶ If $y_1(x) = e^{ax}$ and $y_2(x) = e^{bx}$, then

$$W(y_1, y_2) = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = (b-a)e^{(a+b)x}.$$

- ▶ If $y_1(x) = e^{ax}$ and $y_2(x) = xe^{ax}$, then

$$W(y_1, y_2) = \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & (ax+1)e^{ax} \end{vmatrix} = e^{2ax}.$$

- ▶ If $y_1(x) = e^{\alpha x} \cos \beta x$ and $y_2(x) = e^{\alpha x} \sin \beta x$, then

$$W(y_1, y_2) = \begin{vmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ e^{\alpha x}(\alpha \cos \beta x - \beta \sin \beta x) & e^{\alpha x}(\alpha \sin \beta x + \beta \cos \beta x) \end{vmatrix} = \beta e^{2\alpha x}.$$

Example 5.11 Find the general solution for the equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-x}.$$

Solution. Consider the associated homogeneous equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$$

The characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0 \implies \lambda = -2 \text{ or } -3.$$

The general solution for the associated equation is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{-3x}.$$

Now

$$v_1(x) = - \int \frac{y_2(x) R(x)}{W(y_1, y_2)} dx = - \int \frac{e^{-3x} \cdot e^{-x}}{-e^{-5x}} dx = \int e^x dx = e^x$$

$$v_2(x) = \int \frac{y_1(x) R(x)}{W(y_1, y_2)} dx = \int \frac{e^{-2x} \cdot e^{-x}}{-e^{-5x}} dx = - \int e^{2x} dx = -\frac{1}{2} e^{2x}$$

By variation of parameters, the general solution is

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-2x} + c_2 e^{-3x} + e^x \cdot e^{-2x} - \frac{1}{2} e^{2x} \cdot e^{-3x} = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{2} e^{-x}.$$

Example 5.12 Find the general solution for the equation

$$y'' + y = \csc x.$$

Solution. The characteristic equation is

$$\lambda^2 + 1 = 0 \implies \lambda = \pm i.$$

Hence the two linearly independent solutions of the associated homogeneous equation are $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Thus, $y_1'(x) = -\sin x$ and $y_2'(x) = \cos x$. Also $W(y_1, y_2) = 1$. Now,

$$v_1(x) = - \int \frac{(\sin x)(\csc x)}{1} dx = - \int 1 dx = -x$$

$$v_2(x) = \int \frac{(\cos x)(\csc x)}{1} dx = \int \cot x dx = -\ln |\csc x|.$$

By variation of parameters,

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x) = -x \cos x - (\ln |\csc x|) \sin x$$

is a particular solution and the general solution is

$$y(x) = y_h(x) + y_p(x) = c_1 \cos x + c_2 \sin x - x \cos x - (\ln |\csc x|) \sin x.$$

Example 5.13 Find the general solution for the equation

$$y'' + 2y' + 2y = \sin x.$$

Solution. The characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i.$$

Hence the general solution for the associated homogeneous equation is

$$y_h(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x.$$

Consider

$$v_1(x) = - \int \frac{y_2(x) R(x)}{W(y_1, y_2)} dx = - \int \frac{e^{-x} \sin x \sin x}{e^{-2x}} dx = - \int e^x \sin^2 x dx = ?$$

$$v_2(x) = \int \frac{y_1(x) R(x)}{W(y_1, y_2)} dx = \int \frac{e^{-x} \cos x \sin x}{e^{-2x}} dx = \int e^x \cos x \sin x dx = ?$$

By variation of parameters, the general solution is

$$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x - \left(\int e^x \sin^2 x dx \right) \cdot e^{-x} \cos x \\ + \left(\int e^x \cos x \sin x dx \right) \cdot e^{-x} \sin x. \text{ (please evaluate yourself !!)}$$

Alternative method

Consider $y_p(x) = A \cos x + B \sin x$. Then

$$y'' + 2y' + 2y = (-A \cos x - B \sin x) + 2(-A \sin x + B \cos x) + 2(A \cos x + B \sin x) \\ = (A + 2B) \cos x + (B - 2A) \sin x \stackrel{?}{=} \sin x$$

Solve the system $\begin{cases} A + 2B = 0 \\ -2A + B = 1 \end{cases}$, we have $A = -\frac{2}{5}$ and $B = \frac{1}{5}$. Then the particular solution is

$$y_p(x) = -\frac{2}{5} \cos x + \frac{1}{5} \sin x.$$

Hence the general solution is

$$y(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x + \left(-\frac{2}{5} \cos x + \frac{1}{5} \sin x \right).$$

If $R(x)$	Try $y_p(x)$
$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$	$c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$
e^{ax}	ce^{ax}
$\cos bx$ and/or $\sin bx$	$c_1 \cos bx + c_2 \sin bx$
$e^{ax} \cos bx$ and/or $e^{ax} \sin bx$	$c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx$

If $y_p = y_h \dots$

- ▶ If the above choice of particular solution has already appeared in the solution of the associated homogeneous equation, **one may multiply x to the solution.**
- ▶ If the modified solution still appears in the homogeneous solution, one may multiply by x^2 instead.

Example 5.14 Find the general solution for the equation

$$y'' + 2y' + 2y = x^2 + 1$$

Solution. The characteristic equation is

$$\lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = -1 \pm i.$$

Hence the general solution for the associated homogeneous equation is

$$y_h(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x.$$

Consider $y_p(x) = Ax^2 + Bx + C$. Then

$$\begin{aligned} y'' + 2y' + 2y &= (2A) + 2(2Ax + B) + 2(Ax^2 + Bx + C) \\ &= 2Ax^2 + (4A + 2B)x + (2A + 2B + 2C) \stackrel{?}{=} x^2 + 1 \end{aligned}$$

Take $A = \frac{1}{2}$, $B = -1$ and $C = 1$. Then the general solution is

$$y(x) = y_h(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x + \left(\frac{1}{2} x^2 - x + 1 \right).$$

Linearity principle

Suppose $y_r(x)$ is a particular solution of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

and $y_s(x)$ is a particular solution of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = S(x).$$

Then $y_r(x) + y_s(x)$ is a particular solution of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) + S(x).$$

Proof.

$$\begin{aligned} & \frac{d^2}{dx^2}(y_r + y_s) + P\frac{d}{dx}(y_r + y_s) + Q(y_r + y_s) \\ = & \left(\frac{d^2y_r}{dx^2} + P\frac{dy_r}{dx} + Qy_r \right) + \left(\frac{d^2y_s}{dx^2} + P\frac{dy_s}{dx} + Qy_s \right) \\ = & R(x) + S(x). \end{aligned}$$

Linearity principle

Example Find the general solution for the equation

$$y'' + 2y' + 2y = \sin x + x^2 + 1.$$

Solution. The general solution for the associated homogeneous equation is

$$y_h(x) = c_1e^{-x} \cos x + c_2e^{-x} \sin x.$$

By the previous examples

$$y_r(x) = \left(-\frac{2}{5} \cos x + \frac{1}{5} \sin x \right) \quad \text{and} \quad y_s(x) = \left(\frac{1}{2}x^2 - x + 1 \right)$$

are particular solutions of the DEs

$$y'' + 2y' + 2y = \sin x \quad \text{and} \quad y'' + 2y' + 2y = x^2 + 1.$$

By linearity principle, the general solution is

$$y(x) = c_1e^{-x} \cos x + c_2e^{-x} \sin x + \left(-\frac{2}{5} \cos x + \frac{1}{5} \sin x \right) + \left(\frac{1}{2}x^2 - x + 1 \right).$$

Consider a system of two linear differential equations

$$\begin{cases} x'(t) = 2x(t) + 2y(t) \\ y'(t) = x(t) + 3y(t) \end{cases}$$

The first DE implies

$$x(t) = y'(t) - 3y(t) \implies x'(t) = y''(t) - 3y'(t).$$

Substitute it into the second DE

$$y''(t) - 3y'(t) = 2(y'(t) - 3y(t)) + 3y(t) \implies y''(t) - 5y'(t) + 4y(t) = 0.$$

The general solution of the 2nd order DE is

$$y(t) = c_1 e^t + c_2 e^{4t} \implies x(t) = -2c_1 e^t + c_2 e^{4t}.$$

Let $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Then $X'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ and the system can be rewritten as

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{or equivalently} \quad X'(t) = AX(t) \quad \text{with} \quad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}.$$

The general solution can also be rewritten as

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2c_1 e^t + c_2 e^{4t} \\ c_1 e^t + c_2 e^{4t} \end{bmatrix} = c_1 e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Notice that the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$ with

eigenspaces $\text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ respectively.

In general, if A has an eigenvalue λ with a corresponding eigenvector \mathbf{v} . Then $X(t) = e^{\lambda t} \mathbf{v}$ is a solution of the system $X' = AX$.

Proof. Let $X(t) = e^{\lambda t} \mathbf{v}$. Then

$$X'(t) = \lambda e^{\lambda t} \mathbf{v} = e^{\lambda t} (\lambda \mathbf{v}) = e^{\lambda t} A \mathbf{v} = A(e^{\lambda t} \mathbf{v}) = AX(t).$$

System of homogeneous DEs

If $X_1(t), \dots, X_n(t)$ are n linearly independent solutions of the $n \times n$ system of DEs

$$X'(t) = AX(t)$$

on an open interval, then every solution $X(t)$ of the system can be expressed as

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + \dots + c_n X_n(t).$$

Example 5.15 Find the general solution for the system of DEs

$$X'(t) = AX(t) \quad \text{with} \quad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}.$$

Solution. First, A has eigenvalues

▶ $\lambda_1 = 1$ with eigenspace $\text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$,

▶ $\lambda_2 = 4$ with eigenspace $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

[1]: `A = Matrix([[2,2], [1,3]]); A`

[1]: $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

[2]: `A.eigenvects()`

Then the general solution is

$$X(t) = c_1 e^t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

[2]: [(1,
1,
[Matrix([
[-2],
[1]])]),
(4,
1,
[Matrix([
[1],
[1]])])]

Example 5.16 Find the general solution for the system of DEs

$$X'(t) = AX(t) \quad \text{with} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix}.$$

Solution. First, A has eigenvalues

▶ $\lambda_1 = -2$ with eigenspace $\text{Span} \left\{ \begin{bmatrix} -\frac{4}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix} \right\}$,

▶ $\lambda_2 = -1$ with eigenspace $\text{Span} \left\{ \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix} \right\}$,

▶ $\lambda_3 = 2$ with eigenspace $\text{Span} \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

[3]: `A = Matrix([[1,1,1], [2,1,-1], [-8,-5,-3]]); A`

[3]: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{bmatrix}$

[4]: `A.eigenvects()`

[4]: [(-2,
1,
[Matrix([
[-4/7],
[5/7],
[1]])]),
(-1,
1,
[Matrix([
[-3/2],
[2],
[1]])]),
(2,
1,
[Matrix([
[0],
[-1],
[1]])])]

Then the general solution is

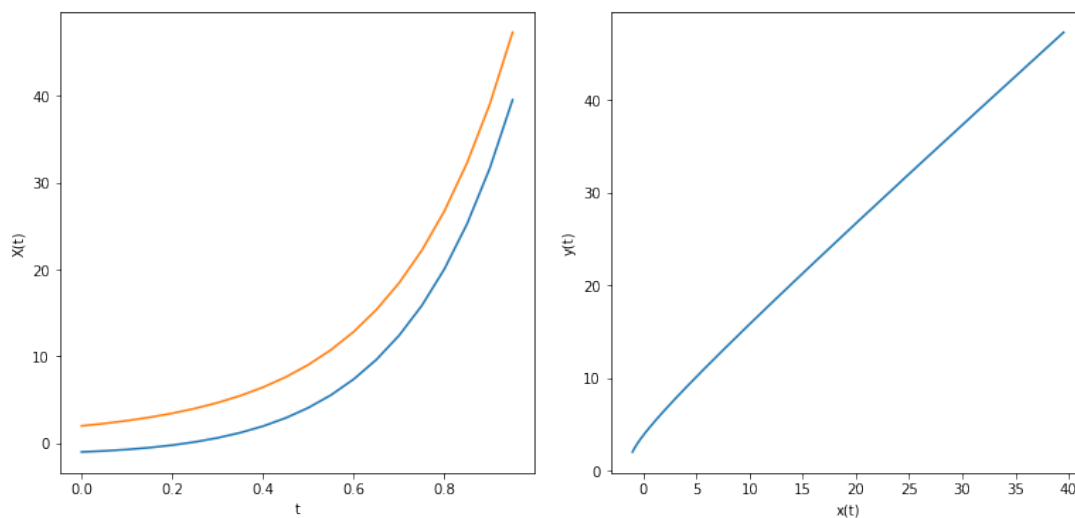
$$X(t) = c_1 e^{-2t} \begin{bmatrix} -\frac{4}{7} \\ \frac{5}{7} \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -\frac{3}{2} \\ 2 \\ 1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

```
[1]: import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

# define the function f(t,X)
def f(t,X):
    x = X[0]
    y = X[1]
    dx = 2*x+2*y
    dy = x+3*y
    dX = np.array([dx, dy])
    return dX

# Solve the ODE numerically
sol = solve_ivp(f, [0, 1], np.array([-1,2]), t_eval=np.arange(0,1,0.05))

# Plot the solution
plt.figure(figsize = (6, 6))
for y in sol.y:
    plt.plot(sol.t,y,label=str(y))
plt.xlabel('t')
plt.ylabel('X(t)')
plt.show()
plt.figure(figsize = (6, 6))
plt.plot(sol.y[0],sol.y[1])
plt.xlabel('x(t)')
plt.ylabel('y(t)')
plt.show()
```



Suppose for a particular commodity, the demand and supply functions are

$$Q_d = a - bP \quad a, b > 0$$

$$Q_s = -c + dP \quad c, d > 0$$

where $P = P(t)$ is the unit price of commodity (over time).

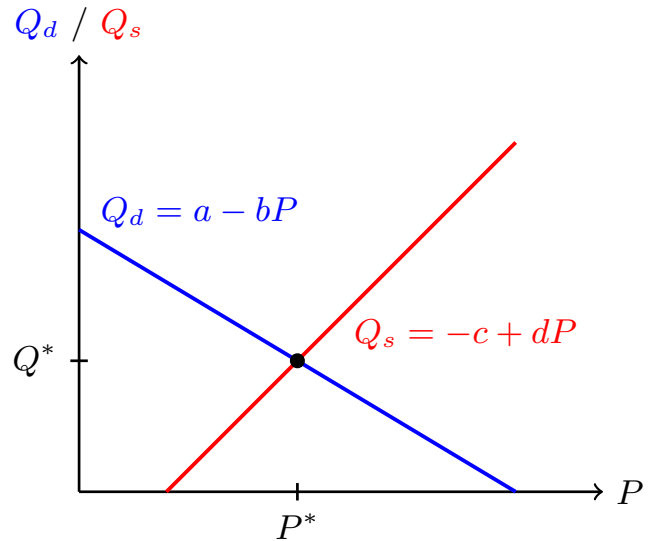
The standard assumption in a supply-demand model is that equilibrium occurs in the market if and only if the market is clear, i.e., the excess demand is zero ($Q_d - Q_s = 0$).

Then the **market-clearing equilibrium price** should be

$$P^* = \frac{a + c}{b + d}.$$

The equilibrium quantity $Q^* = Q_d^* = Q_s^*$ is

$$Q^* = \frac{ad - bc}{b + d}.$$



If the price in the market is not equal to this equilibrium price P^* , then due to market forces, we expect the price will be adjusted over time. One may assume that the rate of price change depends on the **excess demand over supply** $Q_d - Q_s$.

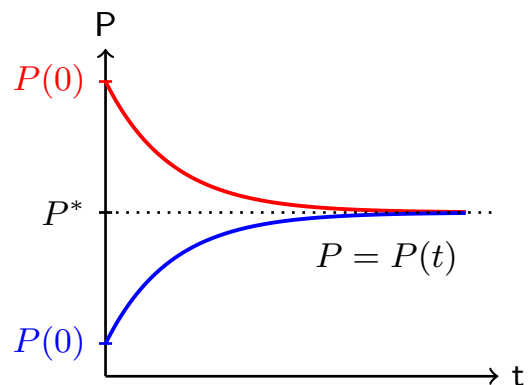
For example, suppose the rate of price change is directly proportional to the excess demand over supply, i.e.,

$$\frac{dP}{dt} = k(Q_d - Q_s) = k(a + c) - k(b + d)P,$$

which is a separable DE. Then the solution is

$$P(t) = (P(0) - P^*)e^{-(b+d)kt} + P^*.$$

When $t \rightarrow \infty$, $P(t) \rightarrow P^*$. That is, the price tends to P^* in long run, which is called the **intertemporal equilibrium**.



Remark: $\frac{dy}{dx} = ay + b \implies y(x) = \left(y(0) + \frac{b}{a}\right)e^{ax} - \frac{b}{a}.$

Example 5.17 Consider the market model

$$Q_d = -5P + 20 \quad Q_s = 3P - 4 \quad \text{and} \quad \frac{dP}{dt} = \frac{1}{5}(Q_d - Q_s).$$

Find the expressions for $P(t)$, $Q_d(t)$, $Q_s(t)$ if $P(0) = 2$.

Solution. Notice that

$$\frac{dP}{dt} = \frac{1}{5}(Q_d - Q_s) = -\frac{8}{5}P + \frac{24}{5} \quad \text{with} \quad P(0) = 2.$$

The particular solution is

$$P(t) = -e^{-\frac{8t}{5}} + 3.$$

Then

$$Q_d(t) = 5e^{-\frac{8t}{5}} + 5 \quad \text{and} \quad Q_s(t) = -3e^{-\frac{8t}{5}} + 5.$$

Example 5.18 The supply and demand functions for a commodity are

$$Q_s = \frac{2}{3}P - 4 \quad \text{and} \quad Q_d = 20 - 2P.$$

Assume that the initial price is $P(0) = 6$, and the price adjusts over time according to the equation $P' = (Q_d - Q_s)^3$, find $P(t)$.

Solution. Notice that

$$\frac{dP}{dt} = (Q_d - Q_s)^3 = \left(24 - \frac{8}{3}P\right)^3 \quad \text{with} \quad P(0) = 6.$$

The particular solution is

$$P(t) = 9 - 3 \left(\frac{3}{3 + 1024t} \right)^{\frac{1}{2}}.$$

In this case, the price tends to 9 when $t \rightarrow \infty$.

Example 5.19 The supply and demand functions for a commodity are

$$Q_d = a - bP - e \frac{dP}{dt} \quad \text{and} \quad Q_s = -c + dP, \quad a, b, c, d, e > 0.$$

- (a) Assuming that the rate of change of price over time is directly proportional to the excess demand over supply, find $P(t)$.
- (b) What are the intertemporal equilibrium price and the market-clearing equilibrium price respectively?

Solution. Notice that

$$\frac{dP}{dt} = k(Q_d - Q_s) = -ke \frac{dP}{dt} - k(b+d)P + k(a+c) \quad \text{for some } k > 0.$$

Then

$$\frac{dP}{dt} = -\frac{k(b+d)}{1+ke}P + \frac{k(a+c)}{1+ke} \implies P(t) = (P(0) - P^*)e^{-\frac{(b+d)kt}{1+ke}} + P^*,$$

where $P^* = \frac{a+c}{b+d}$. Therefore, $P(t) \rightarrow P^*$ when $t \rightarrow \infty$. Hence, the intertemporal equilibrium price is P^* .

For the market-clearing equilibrium,

$$a - bP - e \frac{dP}{dt} = Q_d = Q_s = -c + dP \implies \frac{dP}{dt} = -\frac{b+d}{e}P + \frac{a+c}{e}.$$

The solution is

$$P(t) = (P(0) - P^*)e^{-\frac{(b+d)t}{e}} + P^*.$$

Thus, $P(t) \rightarrow P^*$ when $t \rightarrow \infty$. Hence, the market-clearing equilibrium is also P^* .

Example 5.20 Suppose that consumer demand Q_d depends upon the price-trend according to the equation

$$Q_d = 20 - P - 4\frac{dP}{dt} + \frac{d^2P}{dt^2}.$$

If the supply function is $Q_s = 4P - 4$, write down the condition for equilibrium, and determine the equilibrium price $P(t)$ given that $P(0) = 5$ and $P'(0) = 4$.

Solution. The condition for equilibrium is

$$20 - P - 4\frac{dP}{dt} + \frac{d^2P}{dt^2} = Q_d = Q_s = 4P - 4 \implies \frac{d^2P}{dt^2} - 4\frac{dP}{dt} - 5P = -24.$$

Solving the DP yields,

$$P(t) = c_1e^{5t} + c_2e^{-t} + \frac{24}{5}.$$

With $P(0) = 5$ and $P'(0) = 4$, $c_1 = \frac{7}{10}$ and $c_2 = -\frac{1}{2}$ and hence

$$P(t) = \frac{7}{10}e^{5t} - \frac{1}{2}e^{-t} + \frac{24}{5}.$$

Therefore, $P(t) \rightarrow \infty$ when $t \rightarrow \infty$.

Solow Growth Model

Let K be the capital, L the labour and Y the production output of an economy, as functions of time. Suppose that Q can be expressed as function of K and L , i.e.,

$$Y = f(K, L)$$

with the following assumptions.

- ▶ **Constant returns to scale**

$$f(\lambda K, \lambda L) = \lambda f(K, L) \quad \text{any constant } \lambda.$$

- ▶ **Constant proportion of Y is invested**

$$\frac{dK}{dt} = sY \quad s > 0 \text{ is the proportionality constant.}$$

- ▶ **Labour force grows exponentially**

$$\frac{dL}{dt} = \lambda L \quad \lambda > 0 \text{ is the growth rate.}$$

Then

$$Y = f(K, L) = f\left(L\frac{K}{L}, L \cdot 1\right) = Lf\left(\frac{K}{L}, 1\right) = L\phi(k) \quad \text{where } k = \frac{K}{L},$$

where k can be viewed as the ratio of capital to labour. Here $\phi(k) = f(k, 1)$.

Then

$$\frac{dK}{dt} = sY = sL\phi(k).$$

Also

$$K = kL \implies \frac{dK}{dt} = L\frac{dk}{dt} + k\frac{dL}{dt} \implies sL\phi(k) = L\frac{dk}{dt} + k\lambda L.$$

Therefore,

$$\frac{dk}{dt} = s\phi(k) - \lambda k$$

is the fundamental equation of the [Solow Growth Model](#).

Consider the Cobb-Douglas function

$$f(K, L) = K^a L^{1-a} \quad a \neq 0, 1.$$

Then $\phi(k) = f(k, 1) = k^a$ and so

$$\frac{dk}{dt} = sk^a - \lambda k.$$

Let $y = k^{1-a}$. Then

$$\frac{dy}{dt} = (1-a)k^{-a}\frac{dk}{dt} = (1-a)k^{-a}(sk^a - \lambda k) = (1-a)(s - \lambda y),$$

which is a separable DE. Then the general solution is

$$(k(t))^{1-a} = y(t) = \left(y(0) - \frac{s}{\lambda}\right) e^{-(1-a)\lambda t} + \frac{s}{\lambda}.$$

Therefore, $k(t) \rightarrow \left(\frac{s}{\lambda}\right)^{\frac{1}{1-a}}$ when $t \rightarrow \infty$.

Example 5.21 Suppose the Cobb-Douglas function in the Solow growth model has the form

$$Y = f(K, L) = K^{\frac{1}{3}}L^{\frac{2}{3}}$$

with the proportionality constant $s = \frac{1}{5}$ and the growth rate $\lambda = \frac{1}{10}$. If $K(0) = 8L(0)$, find $k(t)$.

Solution. Let $k = \frac{K}{L}$. Then $k(0) = 8$.

$$\frac{dk}{dt} = \frac{1}{5}k^{\frac{1}{3}} - \frac{1}{10}k.$$

Set $y = k^{\frac{2}{3}}$. Then the equation becomes

$$\frac{dy}{dt} = \frac{2}{15} - \frac{1}{15}y \quad \text{with} \quad y(0) = 4$$

and the solution is

$$y(t) = 2e^{-\frac{t}{15}} + 2 \quad \implies \quad k(t) = \left(2e^{-\frac{t}{15}} + 2\right)^{\frac{3}{2}}.$$