Topic 3 - Eigenvalue and Diagonalization

AMA3724 Further Mathematical Methods(2024/25 Semester 1)
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- Eigenvalues and Eigenvectors
- Similarity
- Diagonalization
- Jordan Canonical Form

Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

1. An **eigenvector** of an $n \times n$ matrix A is a **nonzero** vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x} \quad \text{for some scalar } \lambda.$$

2. The scalar λ is called an **eigenvalue** of A. We also say $\mathbf{x}$ is an **eigenvector** corresponding to λ.

Example Let $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

$$\begin{aligned}
A\mathbf{x} &= \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3\mathbf{x}.\\
A\mathbf{y} &= \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -21 \end{bmatrix} = -7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -7\mathbf{y}.\\
A\mathbf{z} &= \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ -9 \end{bmatrix} \neq \begin{bmatrix} 3\lambda \\ 3\lambda \end{bmatrix} = \lambda\mathbf{z}.
\end{aligned}$$

Eigenvalues and Eigenvectors

Therefore,

- $\lambda = 3$ is an eigenvalue of A and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to 3, and
- $\lambda = -7$ is another eigenvalue of A and $\mathbf{y} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to -7.

Example Let $A = \begin{bmatrix} 4 & 0 & -1 & -2 \\ -1 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$$A\mathbf{x} = \begin{bmatrix} -4 \\ 4 \\ 0 \\ 0 \end{bmatrix} = 4\mathbf{x}, \quad A\mathbf{y} = \begin{bmatrix} 4 \\ 0 \\ -8 \\ 4 \end{bmatrix} = 4\mathbf{y}, \quad \text{and} \quad A\mathbf{z} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = 2\mathbf{z}.$$

- $\lambda = 4$ is an eigenvalue of A and $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ are two linearly independent eigenvectors corresponding to 4, and
- $\lambda = 2$ is another eigenvalue of A and $\mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to 2.

Eigenvalues and Eigenvectors

Characteristic equation

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the **characteristic equation (polynomial)**

$$p_A(\lambda) = \det(A - \lambda I_n) = 0.$$

Solution.

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A \quad &\iff \quad A\mathbf{x} = \lambda\mathbf{x} \text{ for some nonzero } \mathbf{x} \in \mathbb{R}^n \\ &\iff \quad A\mathbf{x} = \lambda I_n\mathbf{x} \text{ for some nonzero } \mathbf{x} \in \mathbb{R}^n \\ &\iff \quad (A - \lambda I_n)\mathbf{x} = \mathbf{0} \text{ has a nontrivial solution} \\ &\iff \quad \mathbf{x} \in \operatorname{Nul}(A - \lambda I_n) \text{ for some nonzero } \mathbf{x} \in \mathbb{R}^n \\ &\iff \quad (A - \lambda I_n) \text{ is singular} \\ &\iff \quad \det(A - \lambda I_n) = 0. \end{aligned}$$

Eigenspace

The eigenspace $E_\lambda(A)$ of A associated with eigenvalue λ is the set of all vectors satisfying $A\mathbf{x} = \lambda\mathbf{x}$ i.e.,

$$E_\lambda(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}.$$

The set $E_\lambda(A)$ contains all vectors in $\operatorname{Nul}(A - \lambda I_n)$. Thus, every nonzero vector in $E_\lambda(A)$ is an eigenvector corresponding to the eigenvalue λ.

Eigenvalues and Eigenvectors

Example 3.1 Find the eigenvalues and all eigenvectors corresponding to each of the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Solution.

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I_2) &= \det\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} \\ &= (2-\lambda)(-6-\lambda) - 9 = \lambda^2 + 4\lambda - 21. \end{aligned}$$

As we need to solve the characteristic equation $\det(A - \lambda I_2) = 0$, which is equivalent to

$$\lambda^2 + 4\lambda - 21 = 0 \quad \Longrightarrow \quad (\lambda - 3)(\lambda + 7) = 0 \quad \Longrightarrow \quad \lambda = 3 \text{ or } -7.$$

Thus, the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -7$.

Eigenvalues and Eigenvectors

For $\lambda_1 = 3$, solve the matrix equation $(A - 3I_2)\mathbf{x} = \mathbf{0}$. Then

$$E_3(A) = \operatorname{Nul}(A - 3I_2) = \operatorname{Span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}.$$

For $\lambda_2 = -7$, solve the matrix equation $(A - (-7)I_2)\mathbf{x} = \mathbf{0}$. Then

$$E_{-7}(A) = \operatorname{Nul}(A + 7I_2) = \operatorname{Span}\left\{\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}.$$

Therefore, A has

- eigenvalue $\lambda_1 = 3$ with eigenspace $E_3(A) = \operatorname{Span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$, and
- eigenvalue $\lambda_2 = -7$ with eigenspace $E_{-7}(A) = \operatorname{Span}\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$.

Eigenvalues and Eigenvectors

Example 3.2 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find the eigenvalues and all eigenvectors corresponding to each of the eigenvalues.

Solution.

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I_3) &= \begin{vmatrix} 4 - \lambda & -1 & 6 \\ 2 & 1 - \lambda & 6 \\ 2 & -1 & 8 - \lambda \end{vmatrix} \\ &= -\lambda^3 + 13\lambda^2 - 40\lambda + 36 = -(\lambda - 2)(\lambda - 2)(\lambda - 9). \end{aligned}$$

Therefore, the eigenvalues of A are 9 and 2.

For $\lambda_1 = 9$, this is equivalent to solve the system $(A - 9I_3)\mathbf{x} = \mathbf{0}$. Then the eigenspace

$$E_9(A) = \mathrm{Nul}\,(A - 9I_3) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda_2 = 2$, this is equivalent to solve the system $(A - 2I_3)\mathbf{x} = \mathbf{0}$. Then the eigenspace

$$E_2(A) = \mathrm{Nul}\,(A - 2I_3) = \mathrm{Span}\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Eigenvalues and Eigenvectors

Example 3.3 Let $A = \begin{bmatrix} 4 & 0 & -1 & -2 \\ -1 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$. Find the eigenvalues and all eigenvectors corresponding to each of the eigenvalues.

Solution.

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I_4) &= \begin{vmatrix} 4 - \lambda & 0 & -1 & -2 \\ -1 & 3 - \lambda & 0 & 1 \\ 0 & 0 & 4 - \lambda & 0 \\ 1 & 1 & -1 & 1 - \lambda \end{vmatrix} \\ &= \lambda^4 - 12\lambda^3 + 52\lambda^2 - 96\lambda + 64 = (\lambda - 4)^2(\lambda - 2)^2. \end{aligned}$$

Therefore, the eigenvalues of A are 4 and 2.

For $\lambda_1 = 4$, this is equivalent to solve the system $(A - 4I_4)\mathbf{x} = \mathbf{0}$. Then the eigenspace

$$E_4(A) = \mathrm{Nul}\,(A - 4I_4) = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda_2 = 2$, this is equivalent to solve the system $(A - 2I_4)\mathbf{x} = \mathbf{0}$. Then the eigenspace

$$E_2(A) = \mathrm{Nul}\,(A - 2I_4) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Eigenvalues and Eigenvectors

Example 3.4 Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$. Find the eigenvalues and all eigenvectors corresponding to each of the eigenvalues.

Solution.

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I_4) &= \begin{vmatrix} 3-\lambda & 1 & -1 \\ 0 & 2-\lambda & 1 \\ 0 & -1 & 2-\lambda \end{vmatrix} \\ &= \lambda^3 - 7\lambda^2 + 17\lambda - 15 = (\lambda - 3)(\lambda^2 - 4x + 5). \end{aligned}$$

Therefore, the eigenvalues of A are 3, $2+i$ and $2-i$.

For $\lambda_1 = 3$, this is equivalent to solve the system $(A - 3I_3)\mathbf{x} = \mathbf{0}$. Then the eigenspace

$$E_3(A) = \operatorname{Nul}(A - 3I_3) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

For $\lambda_2 = 2+i$, this is equivalent to solve the system $(A - (2+i)I_3)\mathbf{x} = \mathbf{0}$. Then the eigenspace

$$E_{2+i}(A) = \operatorname{Nul}(A - (2+i)I_3) = \operatorname{Span}\left\{ \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix} \right\}.$$

Eigenvalues and Eigenvectors

For $\lambda_3 = 2-i$, this is equivalent to solve the system $(A - (2-i)I_3)\mathbf{x} = \mathbf{0}$. Then the eigenspace

$$E_{2-i}(A) = \operatorname{Nul}(A - (2-i)I_3) = \operatorname{Span}\left\{ \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix} \right\}.$$

In fact,

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix} = \begin{bmatrix} -1+2i \\ 1-2i \\ 2+i \end{bmatrix} = \begin{bmatrix} (2+i)i \\ (2+i)(-i) \\ 2+i \end{bmatrix} = (2+i) \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} -1-2i \\ 1+2i \\ 2-i \end{bmatrix} = \begin{bmatrix} (2-i)(-i) \\ (2-i)i \\ 2-i \end{bmatrix} = (2-i) \begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}.$$

Thus, the matrix has complex eigenvalues $2+i$ and $2-i$ with corresponding complex eigenvectors $\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}$ respectively.

Eigenvalues and Eigenvectors

[1]:
```
import sympy as sp; import numpy as np
```

[2]:
```
# Example 3.1
A = sp.Matrix([[2,3],[3,-6]]);A
```

[2]: $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

[3]:
```
# Characteristic polynomial
A.charpoly()
```

[3]: $\mathrm{PurePoly}\left(\lambda^2 + 4\lambda - 21, \lambda, domain = \mathbb{Z}\right)$

[4]:
```
# Factorize the polynomial
sp.factor(A.charpoly().as_expr())
```

[4]: $(\lambda - 3)\,(\lambda + 7)$

[5]:
```
# Eigenvalues of A
A.eigenvals()
```

[5]: {3: 1, -7: 1}

[6]:
```
# Eigenvalues and eigenvectors of A
A.eigenvects()
```

[6]:
```
[(-7,
  1,
  [Matrix([
   [-1/3],
   [   1]])]),
 (3,
  1,
  [Matrix([
   [3],
   [1]])])]
```

Eigenvalues and Eigenvectors

[7]:
```
# Example 3.2
B = sp.Matrix([[4,-1,6], [2,1,6],␣
↪[2,-1,8]]); B
```

[7]: $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$

[8]:
```
# Eigenvalues and eigenvectors of B
B.eigenvects()
```

[8]:
```
[(2,
  2,
  [Matrix([
   [1/2],
   [  1],
   [  0]]),
   Matrix([
   [-3],
   [ 0],
   [ 1]])]),
 (9,
  1,
  [Matrix([
   [1],
   [1],
   [1]])])]
```

[9]:
```
# Example 3.3
C = sp.Matrix([[4,0,-1,-2], [-1,3,0,1],␣
↪[0,0,4,0], [1,1,-1,1]]); C
```

[9]: $\begin{bmatrix} 4 & 0 & -1 & -2 \\ -1 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

[10]:
```
# Eigenvalues and eigenvectors of C
C.eigenvects()
```

[10]:
```
[(2,
  2,
  [Matrix([
   [1],
   [0],
   [0],
   [1]])]),
 (4,
  2,
  [Matrix([
   [-1],
   [ 1],
   [ 0],
   [ 0]]),
   Matrix([
   [ 1],
   [ 0],
   [-2],
   [ 1]])])]
```

Eigenvalues and Eigenvectors

[11]:
```
# Example 3.4
D = sp.Matrix([[3,1,-1],␣
↪[0,2,1], [0,-1,2]]); D
```

[11]:

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

[12]:
```
# Eigenvalues and eigenvectors␣
↪of D
D.eigenvects()
```

[12]:
```
[(3,
  1,
  [Matrix([
   [1],
   [0],
   [0]])]),
 (2 - I,
  1,
  [Matrix([
   [-I],
   [ I],
   [ 1]])]),
 (2 + I,
  1,
  [Matrix([
   [ I],
   [-I],
   [ 1]])])]
```

[13]:
```
# Example 3.2
B = np.array([[4,-1,6], [2,1,6], [2,-1,8]]); B
```

[13]:
```
array([[ 4, -1,  6],
       [ 2,  1,  6],
       [ 2, -1,  8]])
```

[14]:
```
# Eigenvalues and eigenvectors of B @ NumPy
np.linalg.eig(B)
```

[14]:
```
(array([9., 2., 2.]),
 array([[-0.57735027, -0.63269467,  0.13325561],
        [-0.57735027, -0.7699871 , -0.9694382 ],
        [-0.57735027,  0.08256704, -0.20599157]]))
```

[15]:
```
# A 10x10 random matrix
R = np.random.randint(0,4, size=(10,10)); R = R+R.
↪T; R
```

[15]:
```
array([[0, 3, 5, 4, 2, 2, 2, 2, 4, 0],
       [3, 0, 4, 3, 4, 4, 4, 4, 4, 4],
       [5, 4, 2, 1, 0, 6, 6, 3, 6, 4],
       [4, 3, 1, 0, 5, 2, 2, 2, 5, 3],
       [2, 4, 0, 5, 0, 5, 1, 0, 2, 3],
       [2, 4, 6, 2, 5, 6, 6, 1, 3, 3],
       [2, 4, 6, 2, 1, 6, 4, 6, 3, 5],
       [2, 4, 3, 2, 0, 1, 6, 0, 2, 1],
       [4, 4, 6, 5, 2, 3, 3, 2, 0, 6],
       [0, 4, 4, 3, 3, 3, 5, 1, 6, 2]])
```

Eigenvalues and Eigenvectors

[16]:
```
# Eigenvalues and eigenvectors of R @ NumPy
np.linalg.eig(R)
```

[16]:
```
(array([32.17563324,  5.63277259,  3.88026324,  1.7608565 ,  1.01060793,
        -3.21293784, -5.20648718, -7.82496673, -7.35063895, -6.86510279]),
 array([[ 0.23990821,  0.12698866, -0.31397399,  0.59262617, -0.10085464,
          0.09258429,  0.4528656 , -0.0499217 ,  0.50005554,  0.01241008],
        [ 0.32824738,  0.08889643, -0.07734561, -0.02532038,  0.23429608,
         -0.74965133,  0.09324504,  0.08100801, -0.07676899, -0.48959722],
        [ 0.38054555, -0.30367416, -0.07026408,  0.23239732, -0.46619618,
         -0.12890288,  0.03676486,  0.34194048, -0.50214232,  0.31566571],
        [ 0.25200664,  0.46943061, -0.2409439 ,  0.04233546,  0.21516853,
          0.45827272, -0.25336992,  0.4857925 , -0.19897684, -0.24533134],
        [ 0.21909999,  0.53714234,  0.25013033, -0.01500226,  0.29128353,
         -0.07592698,  0.28616397, -0.27485218, -0.29329862,  0.5163472 ],
        [ 0.39167006, -0.01768889,  0.76243963,  0.23029822, -0.05962076,
          0.05952778, -0.32298967,  0.04937532,  0.2990691 , -0.09289477],
        [ 0.39459753, -0.48066242,  0.05561704, -0.1895034 ,  0.30530031,
          0.42073494,  0.38454022, -0.23436875, -0.18937165, -0.25566373],
        [ 0.22379669, -0.32506165, -0.30528602,  0.04529434,  0.53373118,
         -0.11819883, -0.42768526,  0.02357383,  0.23764865,  0.4584884 ],
        [ 0.34134339,  0.16544963, -0.2992427 , -0.08850286, -0.36897361,
          0.04765903, -0.4119943 , -0.65817932, -0.06172418, -0.11649057],
        [ 0.32040244,  0.10236405, -0.05223611, -0.70234897, -0.26206451,
         -0.01589003,  0.18420864,  0.26713627,  0.42508529,  0.18952249]]))
```

Eigenvalues and Eigenvectors

[17]:
```
# Same matrix in SymPy
S = sp.Matrix(R); S
```

[17]:

$$\begin{bmatrix}0 & 3 & 5 & 4 & 2 & 2 & 2 & 2 & 4 & 0\\3 & 0 & 4 & 3 & 4 & 4 & 4 & 4 & 4 & 4\\5 & 4 & 2 & 1 & 0 & 6 & 6 & 3 & 6 & 4\\4 & 3 & 1 & 0 & 5 & 2 & 2 & 2 & 5 & 3\\2 & 4 & 0 & 5 & 0 & 5 & 1 & 0 & 2 & 3\\2 & 4 & 6 & 2 & 5 & 6 & 6 & 1 & 3 & 3\\2 & 4 & 6 & 2 & 1 & 6 & 4 & 6 & 3 & 5\\2 & 4 & 3 & 2 & 0 & 1 & 6 & 0 & 2 & 1\\4 & 4 & 6 & 5 & 2 & 3 & 3 & 2 & 0 & 6\\0 & 4 & 4 & 3 & 3 & 3 & 5 & 1 & 6 & 2\end{bmatrix}$$

[18]:
```
# Eigenvalues and eigenvectors of R @ SymPy
S.eigenvects()
```

[18]:
```
[(CRootOf(_lambda**10 - 14*_lambda**9 - 545*_lambda**8 - 2182*_lambda**7 +
24794*_lambda**6 + 146786*_lambda**5 - 308634*_lambda**4 - 2300298*_lambda**3 +
1472271*_lambda**2 + 9183384*_lambda - 8266448, 0),
  1,
  [Matrix([
   [  6895968363835957306471753587*CRootOf(_lambda**10 - 14*_lambda**9 -
545*_lambda**8 - 2182*_lambda**7 + 24794*_lambda**6 + 146786*_lambda**5 -
308634*_lambda**4 - 2300298*_lambda**3 + 1472271*_lambda**2 + 9183384*_lambda -
8266448, 0)**5/898545375490896937382O2621120 -
5820128746504536882596119*CRootOf(_lambda**10 - 14*_lambda**9 - 545*_lambda**8 -
2182*_lambda**7 + 24794*_lambda**6 + 146786*_lambda**5 - 308634*_lambda**4 -
2300298*_lambda**3 + 1472271*_lambda**2 + 9183384*_lambda - 8266448,
0)**8/898545375490896937382O2621120 +
6298380975215O9919753131*CRootOf(_lambda**10 - 14*_lambda**9 - 545*_lambda**8 -
2182*_lambda**7 + 24794*_lambda**6 + 146786*_lambda**5 - 308634*_lambda**4 -
2300298*_lambda**3 + 1472271*_lambda**2 + 9183384*_lambda - 8266448,
```

Eigenvalues and Eigenvectors

Eigenvalues, Determinant and Diagonal Entries

Given an $n \times n$ matrix $A = [a_{ij}]$ with eigenvalues $\lambda_1, \ldots, \lambda_n$(can be repeated).

1. The characteristic polynomial is equal to

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I_n) &= (-1)^n (\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \\ &= (\lambda_1 - \lambda) \cdot (\lambda_2 - \lambda) \cdots (\lambda_n - \lambda). \end{aligned}$$

2. The product of all eigenvalues of A is equal to the determinant of A, i.e.,

$$\prod_{j=1}^{n} \lambda_j = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det(A).$$

3. The sum of all eigenvalues of A is equal to the sum of all diagonal entries of A, which is also called the Trace of A and denoted by $\operatorname{Tr}(A)$, i.e.,

$$\sum_{j=1}^{n} \lambda_j = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{j=1}^{n} a_{jj} = a_{11} + a_{22} + \cdots + a_{nn} = \operatorname{Tr}(A).$$

4. **Cayley-Hamilton Theorem**:

$$p_A(A) = (-1)^n (A - \lambda_1 I_n) \cdot (A - \lambda_2 I_n) \cdots (A - \lambda_n I_n) = \mathbf{0}.$$

Eigenvalues and Eigenvectors

Proof.

1. Since $\lambda_1, \ldots, \lambda_n$ are roots of the degree n polynomial $\det(A - \lambda I_n)$ and $(-1)^n$ as the leading coefficient (the coefficient of λ^n),

$$\det(A - \lambda I_n) = (-1)^n(\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

2. Substitute $\lambda = 0$ in the equation $p_A(\lambda) = \det(A - \lambda I_n)$,

$$\det A = \det(A - 0I_n) = (-1)^n(0 - \lambda_1)(0 - \lambda_2) \cdots (0 - \lambda_n) = \lambda_1\lambda_2 \cdots \lambda_n.$$

3. Consider the coefficient of λ^{n-1} in the characteristic polynomial,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = (-1)^n\lambda^n + (-1)^n(-\lambda_1 - \lambda_2 - \cdots - \lambda_n)\lambda^{n-1} + \cdots$$

The left hand side has the term $(a_{11} + a_{22} + \cdots a_{nn})(-\lambda)^{n-1}$ and the right hand side has the term $(-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1}$. Thus,

$$a_{11} + a_{22} + \cdots a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Eigenvalues and Eigenvectors

Eigenvalues of Triangular matrix

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof. Suppose $T = [t_{ij}]$ is an $n \times n$ upper triangular matrix, i.e., $t_{ij} = 0$ for all $i > j$. Then $T - \lambda I_n$ is also upper triangular and hence

$$p_T(\lambda) = \det(T - \lambda I_n) = (t_{11} - \lambda)(t_{22} - \lambda) \cdots (t_{nn} - \lambda).$$

Thus, $t_{11}, \ldots, t_{nn}$ are the roots of the characteristic equation $\det(T - \lambda I_n)$. Hence, they are the eigenvalues of T. Similar result holds for lower triangular matrices.

Example Let $A = \begin{bmatrix} 3 & 6 & 8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. Find the eigenvalues of A and B.

Solution. The eigenvalues of A are 3, 0 and 2 and the eigenvalues of B are 4 and 1.

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix with eigenvalue λ.

- The eigenspace $E_\lambda(A)$ of A associated with eigenvalue λ is equal to the null space of $A - \lambda I_n$, i.e., $E_\lambda(A) = \mathrm{Nul}\,(A - \lambda I_n)$.
- The algebraic multiplicity of an eigenvalue λ, denoted by $\mu_A(\lambda)$, is the multiplicity as a root of the characteristic polynomial $\det(A - \lambda I_n) = 0$.
- The geometric multiplicity of an eigenvalue λ, denoted by $\gamma_A(\lambda)$, is the dimension of the eigenspace $E_\lambda(A) = \mathrm{Nul}\,(A - \lambda I_n)$, i.e.,
$$\gamma_A(\lambda) = \dim E_\lambda(A) = \dim \mathrm{Nul}\,(A - \lambda I_n).$$
- The geometric multiplicity is less than or equal to algebraic multiplicity, i.e.,
$$1 \le \gamma_A(\lambda) \le \mu_A(\lambda).$$

		algebraic multiplicity $\mu_A(\lambda)$	geometric multiplicity $\gamma_A(\lambda)$
Example 3.1	$\lambda_1 = 3$	1	1
	$\lambda_2 = -7$	1	1
Example 3.2	$\lambda_1 = 9$	1	1
	$\lambda_2 = 2$	2	2
Example 3.3	$\lambda_1 = 4$	2	2
	$\lambda_2 = 2$	2	1

Eigenvalues and Eigenvectors

Eigenvalues, Determinant and Diagonal Entries

Given an $n \times n$ matrix A with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ and algebraic multiplicities $\mu_A(\lambda_1), \ldots, \mu_A(\lambda_k)$. Then $\mu_A(\lambda_1) + \cdots + \mu_A(\lambda_k) = n$.

1. The characteristic polynomial is equal to
$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I_n) &= (-1)^n (\lambda - \lambda_1)^{\mu_A(\lambda_1)} \cdot (\lambda - \lambda_2)^{\mu_A(\lambda_2)} \cdots (\lambda - \lambda_k)^{\mu_A(\lambda_k)} \\ &= (\lambda_1 - \lambda)^{\mu_A(\lambda_1)} \cdot (\lambda_2 - \lambda)^{\mu_A(\lambda_2)} \cdots (\lambda_k - \lambda)^{\mu_A(\lambda_k)}. \end{aligned}$$
2. The product of all eigenvalues of A (counting algebraic multiplicities) is equal to the determinant of A, i.e.,
$$\prod_{j=1}^{k} \lambda_j^{\mu_A(\lambda_j)} = \lambda_1^{\mu_A(\lambda_1)} \cdot \lambda_2^{\mu_A(\lambda_2)} \cdots \lambda_k^{\mu_A(\lambda_k)} = \det(A).$$
3. The sum of all eigenvalues of A (counting algebraic multiplicities) is equal to the sum of all diagonal entries of A (The Trace of A), i.e.,
$$\sum_{j=1}^{k} \lambda_j \cdot \mu_A(\lambda_j) = \lambda_1 \cdot \mu_A(\lambda_1) + \lambda_2 \cdot \mu_A(\lambda_2) + \cdots + \lambda_k \cdot \mu_A(\lambda_k) = \sum_{j=1}^{n} a_{jj} = \mathrm{Tr}\,(A).$$

Eigenvalues and Eigenvectors

Example 3.5 Let A and B be $n \times n$ matrices. Show that if λ is a nonzero eigenvalue of AB, then λ is also an eigenvalue of BA.

Solution. Suppose λ is a nonzero eigenvalue of AB, then there is an eigenvector $\mathbf{x} \in \mathbb{R}^n$ such that

$$AB\mathbf{x} = \lambda\mathbf{x}.$$

Let $\mathbf{y} = B\mathbf{x}$. Since λ and $\mathbf{x}$ are nonzero, $AB\mathbf{x}$ is nonzero and hence $\mathbf{y} = B\mathbf{x}$ is nonzero. Now

$$AB\mathbf{x} = \lambda\mathbf{x} \quad \Longrightarrow \quad BAB\mathbf{x} = \lambda B\mathbf{x} \quad \Longrightarrow \quad BA\mathbf{y} = \lambda\mathbf{y}.$$

Then λ is an eigenvalue of BA with $\mathbf{y}$ as an eigenvector corresponding to λ.

Exercise What if the assumption nonzero is removed from the statement?

Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$. By Example 3.4, A has

- eigenvalue $\lambda_1 = 3$ with $E_3(A) = \operatorname{Span}\{\mathbf{x}_1\} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$,
- eigenvalue $\lambda_2 = 2 + i$ with $E_{2+i}(A) = \operatorname{Span}\{\mathbf{x}_2\} = \operatorname{Span}\left\{\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}\right\}$,
- eigenvalue $\lambda_3 = 2 - i$ with $E_{2-i}(A) = \operatorname{Span}\{\mathbf{x}_3\} = \operatorname{Span}\left\{\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}\right\}$.

Notice that $\bar{\lambda}_3 = \lambda_2$ and $\bar{\mathbf{x}}_3 = \mathbf{x}_2$. Indeed,

$$A\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \quad \Longrightarrow \quad \overline{A\mathbf{x}_2} = \overline{\lambda_2\mathbf{x}_2} \quad \Longrightarrow \quad \bar{A}\bar{\mathbf{x}}_2 = \bar{\lambda}_2\bar{\mathbf{x}}_2 \quad \Longrightarrow \quad A\bar{\mathbf{x}}_2 = \bar{\lambda}_2\bar{\mathbf{x}}_2,$$

which is equal to $A\mathbf{x}_3 = \lambda_3\mathbf{x}_3$.

Complex eigenvalues and eigenvectors

Let A be an $n \times n$ matrix with real entries. If $\lambda \in \mathbb{C}$ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector in $\mathbb{C}^n$, then the complex conjugate $\bar{\lambda}$ is another eigenvalue of A and $\bar{\mathbf{x}}$ is a corresponding eigenvector.

Similarity

Similarity

For any $n \times n$ matrices A and B,

A **is similar to** B

if there is a nonsingular matrix S such that

$$S^{-1}AS = B.$$

Suppose A is similar to B, then

$$S^{-1}AS = B \quad \Longrightarrow \quad A = SBS^{-1} = \hat{S}^{-1}B\hat{S}.$$

with $\hat{S} = S^{-1}$. Therefore, B is also similar to A. In short, we say simply that **A and B are similar**.

Example Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ and set $S = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$. Then

$$S^{-1}AS = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = B.$$

Thus, A and B are similar.

Similarity

Eigenvalues and Similarity

If two $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same algebraic multiplicities).

Proof. If $B = S^{-1}AS$, then

$$\begin{aligned} p_B(\lambda) = \det(B - \lambda I_n) &= \det(S^{-1}AS - \lambda S^{-1}I_nS) \\ &= \det(S^{-1}(A - \lambda I_n)S) \\ &= \det(S^{-1})\det(A - \lambda I_n)\det(S) \\ &= \det(S^{-1})\det(S)\det(A - \lambda I_n) \\ &= (\det(S))^{-1}\det(S)\det(A - \lambda I_n) \\ &= \det(A - \lambda I_n) = p_A(\lambda). \end{aligned}$$

Diagonalization

Diagonal matrix

A square matrix $D = [d_{ij}]$ is called a **diagonal matrix** if $d_{ij} = 0$ for all $i \neq j$.

Diagonalization

An $n \times n$ matrix A is said to be **diagonalizable** if

$$A = PDP^{-1}$$

for some nonsingular matrix P and some diagonal matrix D.

Example

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \overbrace{\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}}^{P} \overbrace{\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}}^{D} \overbrace{\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}}^{P^{-1}}$$

$$B = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix}}_{P^{-1}}$$

Therefore, A and B are diagonalizable.

Diagonalization

```
[1]: import sympy as sp
```

```
[2]: A = sp.Matrix([[7,2] ,[-4,1]]);A
```

[2]: $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

```
[3]: A.eigenvects()
```

```
[3]:  [(3,
        1,
        [Matrix([
         [-1/2],
         [   1]])]),
       (5,
        1,
        [Matrix([
         [-1],
         [ 1]])])]
```

```
[4]: B = sp.Matrix([[3,2,2,2] ,[2,3,2,2],␣
     ↪[2,2,3,2] ,[2,2,2,3]]); B
```

[4]: $\begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$

```
[5]: B.eigenvects()
```

```
[5]:  [(1,
        3,
        [Matrix([
         [-1],
         [ 1],
         [ 0],
         [ 0]]),
          Matrix([
         [-1],
         [ 0],
         [ 1],
         [ 0]]),
          Matrix([
         [-1],
         [ 0],
         [ 0],
         [ 1]])]),
       (9,
        1,
        [Matrix([
         [1],
         [1],
         [1],
         [1]])])]
```

Diagonalization

Let

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

Now

$$\begin{aligned} AP &= A\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \end{bmatrix} \\ PD &= \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix} \end{aligned}$$

Thus,

$$AP = PD \iff \begin{bmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix}.$$

The equality $AP = PD$ holds if and only if $A\mathbf{v}_j = \lambda_j\mathbf{v}_j$ for $j = 1, \ldots, n$. Thus, λ_j is an eigenvalue of A and $\mathbf{v}_j$ is an eigenvector corresponding to λ_j. Furthermore, P is invertible if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. In this case,

$$AP = PD \implies A = PDP^{-1}.$$

Therefore, A is diagonalizeable if and only if A has n linearly independent eigenvectors. Since P is invertible, the n eigenevectors are linearly independent.

Diagonalization

Diagonalization Theorem

- An $n \times n$ matrix A is diagonalizable if and only only if A has n linearly independent eigenvectors.
- In fact,

  $$A = PDP^{-1} \quad \text{with a diagonal matrix } D \text{ and an invertible } P,$$

  if and only if
  - the columns of P are n linearly independent eigenvectors of A, and
  - the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

$$B = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix}}_{P^{-1}}$$

Diagonalization

Example 3.6 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Solution. Direct computations show that A has

- eigenvalue $\lambda_1 = -2$ with $E_{-2}(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, and
- eigenvalue $\lambda_2 = 1$ with $E_1(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Take

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then P is nonsingular (why?) and

$$AP = PD \quad \Longrightarrow \quad A = PDP^{-1}.$$

Therefore, A is diagonalizable.

Diagonalization

[1]:

```
import sympy as sp
```

[2]:

```
# Example 3.6
A = sp.Matrix([[1,3,3], [-3,-5,-3],␣
↪[3,3,1]]);A
```

[2]:

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

[3]:

```
A.eigenvects()
```

[3]:

```
[(-2,
  2,
  [Matrix([
   [-1],
   [ 1],
   [ 0]]),
   Matrix([
   [-1],
   [ 0],
   [ 1]])]),
 (1,
  1,
  [Matrix([
   [ 1],
   [-1],
   [ 1]])])]
```

[4]:

```
# Diagonalize A
A.diagonalize()
```

[4]:

```
(Matrix([
 [-1, -1,  1],
 [ 1,  0, -1],
 [ 0,  1,  1]]),
 Matrix([
 [-2,  0, 0],
 [ 0, -2, 0],
 [ 0,  0, 1]]))
```

Diagonalization

Example 3.7 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Solution. Direct computations show that A has

- eigenvalue $\lambda_1 = -2$ with
  $$E_{-2}(A) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\},$$
- eigenvalue $\lambda_2 = 1$ with
  $$E_1(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}.$$

Since at most two linearly independent eigenvectors can be found, one cannot construct a nonsingular matrix P Therefore, A is not diagonalizable.

[7]:

```
# Example 3.7
B = sp.Matrix([[2,4,3], [-4,-6,-3],␣
↪[3,3,1]]);B
```

[7]:

$$\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

[8]:

```
B.eigenvects()
```

[8]:

```
[(-2,
  2,
  [Matrix([
   [-1],
   [ 1],
   [ 0]])]),
 (1,
  1,
  [Matrix([
   [ 1],
   [-1],
   [ 1]])])]
```

Diagonalization

[9]:

```
# Diagonalize B
B.diagonalize()
```

```
---------------------------------------------------------------------------
MatrixError                               Traceback (most recent call last)
<ipython-input-8-9e04948e9332> in <module>
      1 # Diagonalize B
----> 2 B.diagonalize()

~/opt/anaconda3/lib/python3.8/site-packages/sympy/matrices/matrices.py in diagonalize(self,␣
↪reals_only, sort, normalize)
    375
    376     def diagonalize(self, reals_only=False, sort=False, normalize=False):
--> 377         return _diagonalize(self, reals_only=reals_only, sort=sort,
    378                 normalize=normalize)
    379

~/opt/anaconda3/lib/python3.8/site-packages/sympy/matrices/eigen.py in _diagonalize(M,␣
↪reals_only, sort, normalize)
    603
    604     if not is_diagonalizable:
--> 605         raise MatrixError("Matrix is not diagonalizable")
    606
    607     if sort:

MatrixError: Matrix is not diagonalizable
```

Diagonalization

Linearly independent eigenvectors (Part I)

If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of an matrix A, then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Proof. Let

$$r = \dim\left(\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}\right).$$

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent. Then $r < k$. By re-order the indices, we may assume that $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent and span Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then there are $c_1, \ldots, c_r$, not all zero, such that

$$\mathbf{v}_{r+1} = c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r.$$

Then

$$\lambda_{r+1}\mathbf{v}_{r+1} = A\mathbf{v}_{r+1} = c_1A\mathbf{v}_1 + \cdots + c_rA\mathbf{v}_r = c_1\lambda_1\mathbf{v}_1 + \cdots + c_r\lambda_r\mathbf{v}_r.$$

Also

$$\lambda_{r+1}\mathbf{v}_{r+1} = c_1\lambda_{r+1}\mathbf{v}_1 + \cdots + c_r\lambda_{r+1}\mathbf{v}_r.$$

Then

$$\mathbf{0} = c_1(\lambda_1 - \lambda_{r+1})\mathbf{v}_1 + \cdots + c_r(\lambda_r - \lambda_{r+1})\mathbf{v}_r.$$

But this implies that $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly dependent, which contradicts to our assumption. So we must have $r = k$ and hence $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Diagonalization

Linearly independent eigenvectors (Part II)

Suppose A has k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. If $\mathbf{v}_{j1}, \ldots, \mathbf{v}_{jn_j}$ are n_j linearly independent eigenvectors corresponding to the eigenvalue λ_j for $j = 1, \ldots, k$. Then the $n_1 + \cdots + n_k$ eigenvectors

$$\mathbf{v}_{11}, \ldots, \mathbf{v}_{1n_1}, \mathbf{v}_{21}, \ldots, \mathbf{v}_{2n_2}, \ldots, \mathbf{v}_{k1}, \ldots, \mathbf{v}_{kn_k}$$

are linearly independent.

Proof. Consider the vector equation

$$\underbrace{c_{11}\mathbf{v}_{11} + \cdots + c_{1n_1}\mathbf{v}_{1n_1}}_{\mathbf{x}_1} + \underbrace{c_{21}\mathbf{v}_{21} + \cdots + c_{2n_2}\mathbf{v}_{2n_2}}_{\mathbf{x}_2} + \cdots + \underbrace{c_{k1}\mathbf{v}_{k1} + \cdots + c_{kn_k}\mathbf{v}_{kn_k}}_{\mathbf{x}_k} = \mathbf{0}.$$

Then $\mathbf{x}_j$ is an eigenvector corresponding to λ_j and

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k = \mathbf{0}.$$

By the previous result, $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent. Thus, their sum is equal to the zero vector if and only if all of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are zero. Thus,

$$c_{j1}\mathbf{v}_{j1} + \cdots + c_{jn_j}\mathbf{v}_{jn_j} = \mathbf{0} \quad j = 1, \ldots, k.$$

and it follows $c_{j1} = \cdots = c_{jn_j} = 0$ for $j = 1, \ldots, k$. So all the $n_1 + \cdots + n_k$ eigenvectors are linearly independent.

Diagonalization

Diagonalization Theorem

Given an $n \times n$ matrix A with distinct eigenvalues $\lambda_1, \dots, \lambda_k$ and algebraic multiplicities $\mu_A(\lambda_1), \dots, \mu_A(\lambda_k)$ and geometric multiplicities $\gamma_A(\lambda_1), \dots, \gamma_A(\lambda_k)$. The matrix A is diagonalizable if and only only if the geometric multiplicity of λ_j is equal to its algebraic multiplicity for all $j = 1, \dots, k$, i.e.,

$$\gamma_A(\lambda_j) = \mu_A(\lambda_j) \quad \text{for all } j = 1, \dots, k.$$

Proof. Notice that $\gamma_A(\lambda_j) \leq \mu_A(\lambda_j)$ for all $j = 1, \dots, k$. Then

$$\sum_{j=1}^{k} \gamma_A(\lambda_j) \leq \sum_{j=1}^{k} \mu_A(\lambda_j) = n.$$

Then A has n linearly independent eigenvectors if and only if the above equality holds. That is, $\gamma_A(\lambda_j) = \mu_A(\lambda_j)$ for all $j = 1, \dots, k$.

Distinct eigenvalues $\Longrightarrow$ diagonalizable

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalization

Example 3.8 Without using any computational devices, show that the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

Solution. If A is diagonalizable, then

$$A = PDP^{-1}$$

But 2 is the only eigenvalue of A, so

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 2I_4.$$

Then

$$A = P(2I_4)P^{-1} = 2I_4,$$

which contradiction arrived. Therefore, A is not diagonalizable.

Diagonalization

Example 3.9 Let $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. Compute

$$(A + 5I_3)^{3724} - (A - I_3)^{3724}.$$

```
[1]: import sympy as sp; import numpy as np
```

```
[2]: # Example 3.9
     A = np.array([[1,3,3] ,[-3,-5,-3], [3,3,1]]);A
```

```
[2]: array([[ 1,  3,  3],
            [-3, -5, -3],
            [ 3,  3,  1]])
```

```
[3]: I = np.eye(3);I
```

```
[3]: array([[1., 0., 0.],
            [0., 1., 0.],
            [0., 0., 1.]])
```

```
[4]: # (A+5I)^2 - (A-I)^2
     np.linalg.matrix_power(A+5*I,2)-np.linalg.matrix_power(A-I,2)
```

```
[4]: array([[ 36.,  36.,  36.],
            [-36., -36., -36.],
            [ 36.,  36.,  36.]])
```

Diagonalization

```
[5]: # (A+5I)^20 - (A-I)^20
     np.linalg.matrix_power(A+5*I,20)-np.linalg.matrix_power(A-I,20)
```

```
[5]: array([[ 3.65615844e+15,  3.65615844e+15,  3.65615844e+15],
            [-3.65615844e+15, -3.65615844e+15, -3.65615844e+15],
            [ 3.65615844e+15,  3.65615844e+15,  3.65615844e+15]])
```

```
[6]: # (A+5I)^3724 - (A-I)^3724
     np.linalg.matrix_power(A+5*I,3724)-np.linalg.matrix_power(A-I,3724)
```

```
/Users/marsze/opt/anaconda3/lib/python3.8/site-
packages/numpy/linalg/linalg.py:662: RuntimeWarning: overflow encountered in
matmul
  z = a if z is None else fmatmul(z, z)
/Users/marsze/opt/anaconda3/lib/python3.8/site-
packages/numpy/linalg/linalg.py:665: RuntimeWarning: invalid value encountered
in matmul
  result = z if result is None else fmatmul(result, z)
/Users/marsze/opt/anaconda3/lib/python3.8/site-
packages/numpy/linalg/linalg.py:662: RuntimeWarning: invalid value encountered
in matmul
  z = a if z is None else fmatmul(z, z)
/Users/marsze/opt/anaconda3/lib/python3.8/site-
packages/numpy/linalg/linalg.py:665: RuntimeWarning: overflow encountered in
matmul
  result = z if result is None else fmatmul(result, z)
```

```
[6]: array([[nan, nan, nan],
            [nan, nan, nan],
            [nan, nan, nan]])
```

Diagonalization

Solution. By Example 3.6, $A = PDP^{-1}$ with

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$\begin{aligned} (A + 5I_3)^{3724} &= P(D + 5I_3)^{3724} P^{-1} \\ &= P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}^{3724} P^{-1} = P \begin{bmatrix} 3^{3724} & 0 & 0 \\ 0 & 3^{3724} & 0 \\ 0 & 0 & 6^{3724} \end{bmatrix} P^{-1} \end{aligned}$$

$$\begin{aligned} (A - I_3)^{3724} &= P(D - I_3)^{3724} P^{-1} \\ &= P \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{3724} P^{-1} = P \begin{bmatrix} (-3)^{3724} & 0 & 0 \\ 0 & (-3)^{3724} & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} \end{aligned}$$

Diagonalization

Then

$$\begin{aligned} (A + 5I_3)^{3724} - (A - I_3)^{3724} &= P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6^{3724} \end{bmatrix} P^{-1} \\ &= \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6^{3724} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\ &= 6^{3724} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\ &= 6^{3724} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Diagonalization

Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$. By Example 3.4, A has

- eigenvalue $\lambda_1 = 3$ with $E_3(A) = \operatorname{Span}\{\mathbf{x}_1\} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$,
- eigenvalue $\lambda_2 = 2 + i$ with $E_{2+i}(A) = \operatorname{Span}\{\mathbf{x}_2\} = \operatorname{Span}\left\{\begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}\right\}$,
- eigenvalue $\lambda_3 = 2 - i$ with $E_{2-i}(A) = \operatorname{Span}\{\mathbf{x}_3\} = \operatorname{Span}\left\{\begin{bmatrix} -i \\ i \\ 1 \end{bmatrix}\right\}$.

Let

$$P = \begin{bmatrix} 1 & i & -i \\ 0 & -i & i \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}.$$

Then

$$A = PDP^{-1} = \begin{bmatrix} 1 & i & -i \\ 0 & -i & i \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & \frac{i}{2} & \frac{1}{2} \\ 0 & -\frac{i}{2} & \frac{1}{2} \end{bmatrix}.$$

Diagonalization

Let

$$\operatorname{Re}\mathbf{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \operatorname{Im}\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_2 = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix} = \operatorname{Re}\mathbf{x}_2 + i\operatorname{Im}\mathbf{x}_2.$$

Set

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Then

$$PDP^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} = A.$$

In general, suppose A has eigenvalue $a + ib$ with eigenvector $\mathbf{x} = \operatorname{Re}\mathbf{x} + i\operatorname{Im}\mathbf{x}$. Then

$$\begin{aligned} A\operatorname{Re}\mathbf{x} + \ iA\operatorname{Im}\mathbf{x} \quad &= \quad A(\operatorname{Re}\mathbf{x} + i\operatorname{Im}\mathbf{x}) \\ &= \quad A\mathbf{x} = (a + ib)(\operatorname{Re}\mathbf{x} + \ \operatorname{Im}\mathbf{x}) \\ &= \quad (a\operatorname{Re}\mathbf{x} - b\operatorname{Im}\mathbf{x}) + i\,(a\operatorname{Im}\mathbf{x} + b\operatorname{Re}\mathbf{x}). \end{aligned}$$

Diagonalization

$$A\begin{bmatrix}\operatorname{Re}\mathbf{x} & \operatorname{Im}\mathbf{x}\end{bmatrix}=\begin{bmatrix}A\operatorname{Re}\mathbf{x} & A\operatorname{Im}\mathbf{x}\end{bmatrix} = \begin{bmatrix}a\operatorname{Re}\mathbf{x}-b\operatorname{Im}\mathbf{x} & a\operatorname{Im}\mathbf{x}+b\operatorname{Re}\mathbf{x}\end{bmatrix}$$

$$= \begin{bmatrix}\operatorname{Re}\mathbf{x} & \operatorname{Im}\mathbf{x}\end{bmatrix}\begin{bmatrix}a & b\\ -b & a\end{bmatrix}.$$

Factorization with real matrices

Suppose an $n\times n$ matrix A with real entries is diagonalizable. Then there exists an invertible matrix P with real entries such that

$$A = P\begin{bmatrix}D_1 & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & D_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \mathbf{0}\\ \mathbf{0} & \cdots & \mathbf{0} & D_k\end{bmatrix}P^{-1},$$

where either (1) $D_j=[\lambda_j]$ is an 1×1 block matrix, or (2) $D_j=\begin{bmatrix}a_j & b_j\\ -b_j & a_j\end{bmatrix}$ is a 2×2 block matrix. Here λ_j is the real eigenvalue of A and $a_j\pm ib_j$ are the complex eigenvalues of A.

Jordan Canonical Form

Jordan block

A Jordan block $J_k(\lambda)$ is a $k\times k$ upper triangular matrix of the form

$$J_k(\lambda)=\begin{bmatrix}\lambda & 1 & 0 & \cdots & 0\\ 0 & \lambda & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \lambda & 1\\ 0 & \cdots & \cdots & 0 & \lambda\end{bmatrix}.$$

In particular,

$$J_1(\lambda)=\begin{bmatrix}\lambda\end{bmatrix},\quad J_2(\lambda)=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix},\quad J_3(\lambda)=\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix},\quad J_4(\lambda)=\begin{bmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & \lambda\end{bmatrix}.$$

Jordan Canonical Form

Jordan Canonical Form

Let A be an $n \times n$ matrix. There is an nonsingular matrix S such that

$$A = SJS^{-1} = S \underbrace{\begin{bmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{n_r}(\lambda_r) \end{bmatrix}}_{J} S^{-1},$$

where $n_1 + \cdots + n_r = n$. Here the eigenvalues $\lambda_1, \ldots, \lambda_r$ are not necessarily distinct.

If $r = n$ and $n_1 = \cdots = n_r = 1$. Then

$$J = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix},$$

and thus, A is diagonalizable.

Reference: R.A. Horn & C.R. Johnson, Matrix Analysis 2nd Edition, Cambridge University Press (2012)

Jordan Canonical Form

Example The matrix $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ has

- eigenvalue $\lambda_1 = -2$ with $\mu_A(-2) = 2$ and $E_{-2}(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$,

  and
- eigenvalue $\lambda_2 = 1$ with $\mu_A(1) = 1$ and $E_1(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

So A is not diagonalizable. But

$$A = SJS^{-1} = \underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{S^{-1}}.$$

Here $n_1 = 2$ and $n_2 = 1$.

Remark: The vector $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is a solution of the equation $(A + 2I_3)\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Jordan Canonical Form

Example The matrix $A = \begin{bmatrix} 4 & 0 & -1 & -2 \\ -1 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$ has

- eigenvalue $\lambda_1 = 4$ with $\mu_A(1) = 2$ and
  $$E_4(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\},$$ and
- eigenvalue $\lambda_2 = 2$ with $\mu_A(2) = 2$ and $E_2(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

So A is not diagonalizable. But

$$A = SJS^{-1} = \underbrace{\begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 0 & 0 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}}_{S^{-1}}.$$

Here $n_1 = 2$ and $n_2 = n_3 = 1$.

Jordan Canonical Form

[1]:
```
import sympy as sp
```

[2]:
```
A = sp.Matrix([[2,4,3], [-4,-6,-3],␣
↪[3,3,1]]);A
```

[2]:
$$\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

[3]:
```
A.eigenvects()
```

[3]:
```
[(-2,
  2,
  [Matrix([
   [-1],
   [ 1],
   [ 0]])]),
 (1,
  1,
  [Matrix([
   [ 1],
   [-1],
   [ 1]])])]
```

[4]:
```
A.jordan_form()
```

[4]:
```
(Matrix([
 [-1, -1,  1],
 [ 1,  0, -1],
 [ 0,  1,  1]]),
 Matrix([
 [-2,  1, 0],
 [ 0, -2, 0],
 [ 0,  0, 1]]))
```

Jordan Canonical Form

[5]:

```
B = sp.Matrix([[4,0,-1,-2], [-1,3,0,1],␣
↪[0,0,4,0], [1,1,-1,1]]);B
```

[5]:

$$\begin{bmatrix} 4 & 0 & -1 & -2 \\ -1 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

[6]:

```
B.eigenvects()
```

[6]:

```
[(2,
  2,
  [Matrix([
   [1],
   [0],
   [0],
   [1]])]),
 (4,
  2,
  [Matrix([
   [-1],
   [ 1],
   [ 0],
   [ 0]]),
   Matrix([
   [ 1],
   [ 0],
   [-2],
   [ 1]])])]
```

[7]:

```
B.jordan_form()
```

[7]:

```
(Matrix([
 [2, 1, -1,  1],
 [0, 1,  1,  0],
 [0, 0,  0, -2],
 [2, 0,  0,  1]]),
 Matrix([
 [2, 1, 0, 0],
 [0, 2, 0, 0],
 [0, 0, 4, 0],
 [0, 0, 0, 4]]))
```