Topic 1 - Vector Space and Subspace

AMA3724 Further Mathematical Methods(2024/25 Semester 1)
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- Linear combination
- Spanning set
- Linear independence
- Vector Space and subspace
- Null space and column space
- Basis and dimension
- Row space
- Rank
- Change of basis

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Vectors

Vectors in $\mathbb{R}^n$

- A $n \times 1$ matrix (with only one column) is called a **column vector** , or simply a **vector**.
- The set of all $n \times 1$ vectors is denoted by $\mathbb{R}^n$.
- The vector whose entries all zero is called the **zero vector** and is denotes by $\mathbf{0}$.

Example

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ -5 \\ 4 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$$\text{(Addition:)} \qquad \mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}$$

$$\text{(Scalar Multiplication:)} \qquad (-3)\mathbf{v} = -3 \cdot \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ 15 \\ 3 \end{bmatrix}$$

$$\text{(Linear Combination:)} \qquad 4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \\ 12 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ 15 \end{bmatrix}$$

Vectors

Algebraic Properties of $\mathbb{R}^n$

For any $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and scalars c and d,

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$,
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$,
3. $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$,
4. $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$, where $-\mathbf{u}=(-1)\mathbf{u}$,
5. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$,
6. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$,
7. $c(d\mathbf{u})=(cd)\mathbf{u}$,
8. $1\mathbf{u}=\mathbf{u}$.

Linear combination

Linear combination

Given vectors $\mathbf{v}_1,\dots,\mathbf{v}_p$ in $\mathbb{R}^n$ and scalars $c_1,\dots,c_p$, the vector $\mathbf{b}$ defined by

$$\mathbf{b}=c_1\mathbf{v}_1+\cdots+c_p\mathbf{v}_p=\sum_{j=1}^{p}c_j\mathbf{v}_j$$

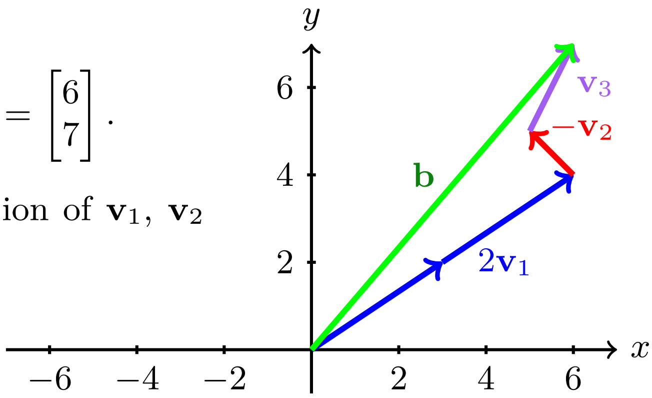
is called a linear combination of $\mathbf{v}_1,\dots,\mathbf{v}_p$ with weights $c_1,\dots,c_p$.

Example Let $\mathbf{v}_1=\begin{bmatrix}3\\2\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{v}_3=\begin{bmatrix}1\\2\end{bmatrix}$.

Take

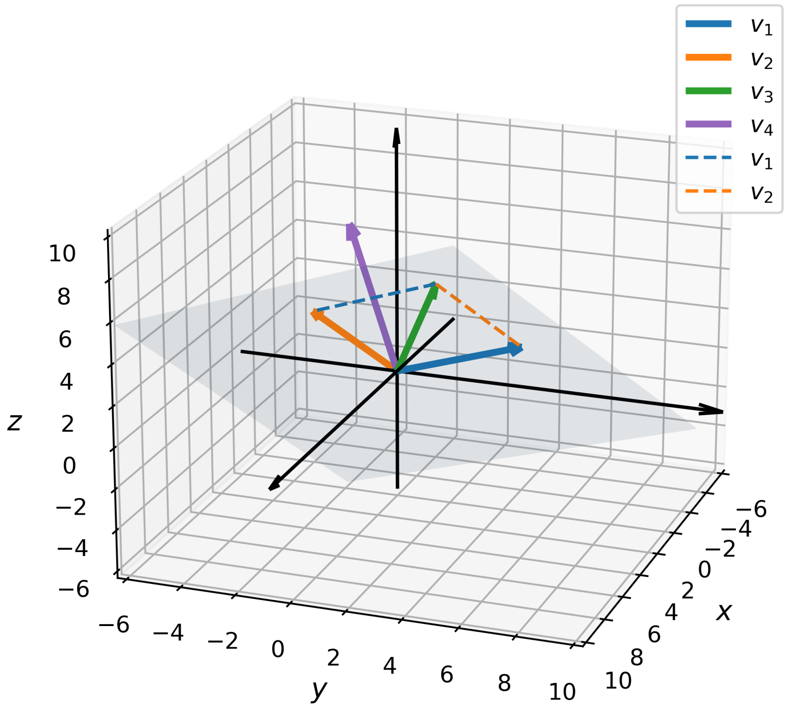
$$\begin{aligned}\mathbf{b} &= 2\mathbf{v}_1+(-1)\mathbf{v}_2+\mathbf{v}_3\\ &= 2\begin{bmatrix}3\\2\end{bmatrix}+(-1)\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}6\\7\end{bmatrix}.\end{aligned}$$

Then the vector $\mathbf{b}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.



Linear combination

Example Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 6 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 4 \\ 8 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}$. Then $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ while $\mathbf{v}_4$ is NOT a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.



Linear combination

[1]:
```python
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

# create the figure
fig = plt.figure(figsize=[10,6],dpi=500)

# add axes
ax = fig.add_subplot(111, projection='3d')
ax.set_xlim(-6, 10)
ax.set_ylim(-6, 10)
ax.set_zlim(-6, 10)
ax.view_init(elev=20, azim=20)

# plot x-axis / y-axis / z-axis
ax.set_xlabel(r'$x$', fontsize='large')
ax.set_ylabel(r'$y$', fontsize='large')
ax.set_zlabel(r'$z$', fontsize='large')
ax.quiver(-6, 0, 0, 18, 0, 0, color='k', arrow_length_ratio=0.05)
ax.quiver(0, -6, 0, 0, 18, 0, color='k', arrow_length_ratio=0.05)
ax.quiver(0, 0, -6, 0, 0, 18, color='k', arrow_length_ratio=0.05)

# plot vectors v_1 v_2 v_3 v_4
ax.quiver(0, 0, 0, 4, 6, 4, color='C0', arrow_length_ratio=0.1, linewidth=3, label='${v_1}$')
ax.quiver(0, 0, 0, 3, -2, 4, color='C1', arrow_length_ratio=0.1, linewidth=3, label='${v_2}$')
ax.quiver(0, 0, 0, 7, 4, 8, color='C2', arrow_length_ratio=0.1, linewidth=3, label='${v_3}$')
ax.quiver(0, 0, 0, 2, -1, 8, color='C4', arrow_length_ratio=0.1, linewidth=3, label='${v_4}$')
ax.quiver(3, -2, 4, 4, 6, 4, color='C0', arrow_length_ratio=0, linestyle = '--', label=r'${v_1}$')
ax.quiver(4, 6, 4, 3, -2, 4, color='C1', arrow_length_ratio=0, linestyle = '--', label=r'${v_2}$')

#plot the plane region
s, t = np.meshgrid(range(-15,15), range(-15,15))
s = s*.1
t = t*.1
X = s*4+t*3
Y = s*6+t*-2
Z = s*4+t*4
ax.plot_surface(X, Y, Z, alpha=0.1)

plt.legend()

plt.show()
```

Linear combination vs Matrix equation

Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then

$$\sum_{j=1}^{n} x_j \mathbf{a}_j = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \iff \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b} \iff A\mathbf{x} = \mathbf{b}.$$

So $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if $A\mathbf{x} = \mathbf{b}$ is consistent.

Spanning Set

Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$. The set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by

$$\mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$$

and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. So a vector $\mathbf{b} \in \mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ when $\mathbf{b}$ can be written in the form

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p \quad \text{with weighting } c_1, \dots, c_p.$$

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Example 1.1 Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ -4 \\ 5 \\ -2 \\ 3 \end{bmatrix}, \mathbf{b}_1 = \begin{bmatrix} 4 \\ 8 \\ -8 \\ 9 \\ 4 \end{bmatrix}, \text{ and } \mathbf{b}_2 = \begin{bmatrix} 4 \\ 8 \\ -8 \\ 9 \\ 5 \end{bmatrix}.$$

Determine whether $\mathbf{b}_1$ and $\mathbf{b}_2$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. That is, whether $\mathbf{b}_1$ and $\mathbf{b}_2$ are in $\mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ or not.

Solution. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -4 \\ -5 & 6 & 5 \\ 3 & -1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$. Consider the matrix equation $A\mathbf{x} = \mathbf{b}_j$ for $j = 1, 2$.

[1]:
```
A = sp.Matrix([[1,2,3],[-2,5,-4],[-5,6,5],[3,-1,-2],[1,2,3]]); A
```

[1]:
$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & -4 \\ -5 & 6 & 5 \\ 3 & -1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

[2]:
```
x1,x2,x3 = sp.symbols('x1 x2 x3')
```

Linear combination

[3]:
```
b1 = sp.Matrix([[4,8,-8,9,4]]).T; b1
```

[3]: $\begin{bmatrix} 4 \\ 8 \\ -8 \\ 9 \\ 4 \end{bmatrix}$

[4]:
```
b2 = sp.Matrix([[4,8,-8,9,5]]).T; b2
```

[4]: $\begin{bmatrix} 4 \\ 8 \\ -8 \\ 9 \\ 5 \end{bmatrix}$

[5]:
```
sp.linsolve((A,b1),x1,x2,x3)
```

[5]: $\{(3,\ 2,\ -1)\}$

[6]:
```
sp.linsolve((A,b2),x1,x2,x3)
```

[6]: $\emptyset$

Therefore, $A\mathbf{x} = \mathbf{b}_1$ is consistent and $A\mathbf{x} = \mathbf{b}_2$ is inconsistent. So $\mathbf{b}_1$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ while $\mathbf{b}_2$ is not. That is $\mathbf{b}_1 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathbf{b}_2 \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In particular,

$$\mathbf{b}_1 = 3\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3.$$

Spanning set

Some simple facts about $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$

Given p vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$. For any scalar c and $j = 1, \ldots, p$,

1. $\mathbf{v}_j \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
2. $c\mathbf{v}_j \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
3. $\mathbf{0} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.
4. If $\mathbf{v}_p$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}$, then

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}, \mathbf{v}_p\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}\}.$$

That is, the set remains unchanged after removing the vector $\mathbf{v}_p$.

Proof. The results hold since for any scalar c and $j = 1, \ldots, p$

$$\begin{aligned} \mathbf{v}_j &= 0\mathbf{v}_1 + \cdots 0\mathbf{v}_{j-1} + 1\mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_p, \\ c\mathbf{v}_j &= 0\mathbf{v}_1 + \cdots 0\mathbf{v}_{j-1} + c\mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_p, \\ \mathbf{0} &= 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_p \end{aligned}$$

Proof. (cont.) It is clear that

$$\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}\} \subseteq \text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{p-1},\mathbf{v}_p\}.$$

Now we **want to show** that

$$\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{p-1},\mathbf{v}_p\} \subseteq \text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}\}.$$

Suppose $\mathbf{v}_p$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}$, then

$$\mathbf{v}_p = d_1\mathbf{v}_1 + \cdots + d_{p-1}\mathbf{v}_{p-1} \quad \text{for some scalars } d_1,\ldots,d_{p-1}.$$

For any $\mathbf{b} \in \text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$,

$$\begin{aligned}
\mathbf{b} &= c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p \\
&= c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} + c_p(d_1\mathbf{v}_1 + \cdots + d_{p-1}\mathbf{v}_{p-1}) \\
&= (c_1 + c_pd_1)\mathbf{v}_1 + \cdots (c_{p-1} + c_pd_{p-1})\mathbf{v}_{p-1}
\end{aligned}$$

Thus, $\mathbf{b}$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}$, i.e., $\mathbf{b} \in \text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}\}$. Hence, $\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_p\} \subseteq \text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}\}$ and $\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_p\} = \text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_{p-1}\}$.

Spanning $\mathbb{R}^m$

A set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** $\mathbb{R}^m$ if every vector $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1,\ldots,\mathbf{v}_p$. Equivalently,

$$\text{Span}\{\mathbf{v}_1,\ldots,\mathbf{v}_p\} = \mathbb{R}^m.$$

Some equivalent conditions

Let A be an $m \times n$ matrix. Then the following statements are equivalent.

1. The columns of A span $\mathbb{R}^m$.
2. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
3. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent.
4. A has pivot positions in every row.

Example Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\-2\\-5\\3\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}2\\5\\6\\-1\\2\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}3\\-4\\5\\-2\\3\end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix}4\\8\\-8\\10\\4\end{bmatrix}, \text{ and } \mathbf{v}_5 = \begin{bmatrix}4\\8\\-8\\9\\5\end{bmatrix}.$$

Then $\text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4,\mathbf{v}_5\} = \mathbb{R}^5$. (Hint: the RREF of $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5\end{bmatrix}$.)

Linearly independent and dependent

Given an set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$. Consider the vector equation

$$\sum_{j=1}^{p} x_j \mathbf{v}_j = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}.$$

- The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be linearly independent if the vector equation has trivial solution only (unique solution). i.e., $\sum_{j=1}^{p} x_j\mathbf{v}_j = \mathbf{0}$ implies $x_1 = \cdots = x_p = 0$.
- The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be linearly dependent if the vector equation has non-trivial solution (infinitely many solutions). i.e., there are scalars $x_1, \dots, x_p$, NOT ALL ZERO, such that $\sum_{j=1}^{p} x_j\mathbf{v}_j = \mathbf{0}$.

Linear independence vs Matrix equation

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Proof. Let $A = \begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix}$. Notice that the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent if and only if the vector equation

$$A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \quad \text{has trivial solution only.}$$

Example 1.2 Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\-5\\3\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}4\\8\\-8\\9\end{bmatrix}$. $\mathbf{v}_3 = \begin{bmatrix}2\\5\\6\\-1\end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix}3\\-4\\5\\-2\end{bmatrix}$.

Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent and find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ if the set is linearly dependent.

Solution. Let

$$A = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{bmatrix} = \begin{bmatrix}1 & 4 & 2 & 3\\-2 & 8 & 5 & -4\\-5 & -8 & 6 & 5\\3 & 9 & -1 & -2\end{bmatrix}.$$

Then the general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\begin{cases} x_1 &= -3x_4\\ x_2 &= x_4\\ x_3 &= -2x_4\\ x_4 & \text{is free}\end{cases}.$$

Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. Furthermore,

$$-3\mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

[1]:
```
A = sp.Matrix([[1,4,2,3], [-2,8,5,-4], [-5,-8,6,5], [3,9,-1,-2]]); A
```

[1]: $\begin{bmatrix}1 & 4 & 2 & 3\\-2 & 8 & 5 & -4\\-5 & -8 & 6 & 5\\3 & 9 & -1 & -2\end{bmatrix}$

[2]:
```
b =sp.zeros(4,1); b
```

[2]: $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

[3]:
```
x1,x2,x3,x4 = sp.symbols('x1 x2 x3 x4')
```

[4]:
```
sp.linsolve((A,b),x1,x2,x3,x4)
```

[4]: $\{(-3x_4,\ x_4,\ -2x_4,\ x_4)\}$

Example 1.3 Determine if the columns of the matrix $A = \begin{bmatrix} 2 & 8 & 5 & 5 \\ 3 & 6 & 5 & 3 \\ 1 & 1 & 0 & 0 \\ 8 & 5 & 4 & 0 \\ 6 & 8 & 1 & 6 \\ 8 & 7 & 3 & 1 \\ 5 & 3 & 7 & 2 \end{bmatrix}$ are linearly independent.

Solution. Row reduce the matrix A as follows.

$$A = \begin{bmatrix} 2 & 8 & 5 & 5 \\ 3 & 6 & 5 & 3 \\ 1 & 1 & 0 & 0 \\ 8 & 5 & 4 & 0 \\ 6 & 8 & 1 & 6 \\ 8 & 7 & 3 & 1 \\ 5 & 3 & 7 & 2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

All variables of $A\mathbf{x} = \mathbf{0}$ are basic variables. Thus, the equation $A\mathbf{x} = \mathbf{0}$ has trivial solution only. So the columns of A are linearly independent.

[5]:
```
A = sp.Matrix([[2,8,5,5],
[3,6,5,3], [1,1,0,0],
[8,5,4,0], [6,8,1,6],
[8,7,3,1], [5,3,7,2]]); A
```

[5]:
$$\begin{bmatrix} 2 & 8 & 5 & 5 \\ 3 & 6 & 5 & 3 \\ 1 & 1 & 0 & 0 \\ 8 & 5 & 4 & 0 \\ 6 & 8 & 1 & 6 \\ 8 & 7 & 3 & 1 \\ 5 & 3 & 7 & 2 \end{bmatrix}$$

[6]:
```
A.rref()
```

[6]:
```
(Matrix([
 [1, 0, 0, 0],
 [0, 1, 0, 0],
 [0, 0, 1, 0],
 [0, 0, 0, 1],
 [0, 0, 0, 0],
 [0, 0, 0, 0],
 [0, 0, 0, 0]]),
 (0, 1, 2, 3))
```


Some facts about linear independence

1. The set of **two** vectors is linearly independent if and only if neither of the vectors is a scalar multiple of other.
2. If a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ contains the zero vector $\mathbf{0}$, then the set is linearly dependent.
3. Any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ is linearly dependent if $n > m$.
4. A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ is a linear combination of the others.

Proof.

1. Suppose the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are scalar multiple, say, $\mathbf{v}_1 = k\mathbf{v}_2$. Then $(-1)\mathbf{v}_1 + k\mathbf{v}_2 = \mathbf{0}$ so $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly dependent.
   Now suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. Then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ for some nonzero scalar c_1 and c_2. Then $\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2$, so $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$.
2. Suppose $\mathbf{v}_p = \mathbf{0}$. Then it is clear that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent since

$$0 \cdot \mathbf{v}_1 + \cdots + 0 \cdot \mathbf{v}_{p-1} + 1 \cdot \mathbf{v}_p = \mathbf{0}.$$

Proof. (cont.)

3. Let $A = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}$. Since $n > m$, the system $A\mathbf{x} = \mathbf{0}$ must have non-trivial solutions. Therefore, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly dependent.

4. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. Then there exist $c_1, \ldots, c_p$, not all zeros, such that $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$. Suppose $c_j \neq 0$. Then

$$\mathbf{v}_j = -\frac{c_1}{c_j}\mathbf{v}_1 - \cdots - \frac{c_{j-1}}{c_j}\mathbf{v}_{j-1} - \frac{c_{j+1}}{c_j}\mathbf{v}_{j+1} - \cdots - \frac{c_p}{c_j}\mathbf{v}_p.$$

Thus, $\mathbf{v}_j$ is a linear combination of the others.
On the other hand, suppose $\mathbf{v}_j$ is a linear combination of the others, say

$$\mathbf{v}_j = k_1\mathbf{v}_1 + \cdots + k_{j-1}\mathbf{v}_{j-1} + k_{j+1}\mathbf{v}_{j+1} + \cdots + k_p\mathbf{v}_p$$

for some scalars k_i. Then

$$k_1\mathbf{v}_1 + \cdots + k_{j-1}\mathbf{v}_{j-1} + (-1)\mathbf{v}_j + k_{j+1}\mathbf{v}_{j+1} + \cdots + k_p\mathbf{v}_p = \mathbf{0}$$

and so $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

Example Determine if the given set is linearly dependent.

1. $\begin{bmatrix}1\\7\\6\\-1\end{bmatrix}, \begin{bmatrix}2\\0\\9\\4\end{bmatrix}, \begin{bmatrix}3\\1\\5\\-2\end{bmatrix}, \begin{bmatrix}4\\1\\8\\3\end{bmatrix}, \begin{bmatrix}1-\\1\\0\\7\end{bmatrix}$

2. $\begin{bmatrix}2\\3\\5\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\\-3\end{bmatrix}$

3. $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$

Solution.

1. Linearly dependent. There are 5 vectors in $\mathbb{R}^4$.
2. Linearly dependent. There is a zero vector.
3. Linearly independent. The two vectors are not scalar multiple of each other.

Example 1.4 Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be linearly independent vectors in $\mathbb{R}^n$ and A be an $n \times n$ nonsingular matrix. Show that $A\mathbf{v}_1, \ldots, A\mathbf{v}_p$ are linearly independent.

Solution. Consider the vector equation

$$x_1 A\mathbf{v}_1 + \cdots + x_p A\mathbf{v}_p = \mathbf{0} \quad \Longrightarrow \quad A(x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p) = \mathbf{0}.$$

Since A is invertible,

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = A^{-1}\mathbf{0} = \mathbf{0}.$$

As $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly independent, we must have $x_1 = \cdots = x_p = 0$. Thus, $A\mathbf{v}_1, \ldots, A\mathbf{v}_p$ are also linearly independent.

Example Consider

$$\overbrace{\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5\end{bmatrix}}^{A} = \begin{bmatrix}1 & 4 & 0 & 2 & -1\\ 3 & 12 & 1 & 5 & 5\\ 2 & 8 & 1 & 3 & 2\\ 5 & 20 & 2 & 8 & 8\end{bmatrix} \sim \begin{bmatrix}1 & 4 & 0 & 2 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Then

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_3 & \mathbf{a}_5 & \mathbf{a}_2 & \mathbf{a}_4\end{bmatrix} = \begin{bmatrix}1 & 0 & -1 & 4 & 2\\ 3 & 1 & 5 & 12 & 5\\ 2 & 1 & 2 & 8 & 3\\ 5 & 2 & 8 & 20 & 8\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 4 & 2\\ 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Notice that

$$\mathbf{a}_2 = 4\mathbf{a}_1 + 0\mathbf{a}_3 + 0\mathbf{a}_5 \quad \text{and} \quad \mathbf{a}_4 = 2\mathbf{a}_1 + (-1)\mathbf{a}_3 + 0\mathbf{a}_5.$$

It follows that $\mathbf{a}_2$ and $\mathbf{a}_4$ are linear combinations of $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$. Then

$$\text{Col}\, A = \text{Span}\,\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\} = \text{Span}\,\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\} = \text{Span}\,\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}.$$

Also $\mathbf{a}_1 = \begin{bmatrix}1\\3\\2\\5\end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix}0\\1\\1\\2\end{bmatrix}$, and $\mathbf{a}_5 = \begin{bmatrix}-1\\5\\2\\8\end{bmatrix}$ are linearly independent.

Vector Space

A set V of objects, called vectors, on which the two operations, called addition and scalar multiplication are well defined if

- The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
- The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.

The set V together with the addition and scalar multiplication is called a vector space if the axioms listed below hold for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
3. There is a zero vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
4. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
5. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
6. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
7. $c(d\mathbf{u})=(cd)\mathbf{u}$.
8. $1\mathbf{u}=\mathbf{u}$.

Examples of Vector Space

1. The set $\mathbb{R}^n$ with the usual addition and scalar multiplication
2. The set of complex numbers $\mathbb{C}$ with the usual addition and scalar multiplication
3. The set of n-tuples of complex numbers $\mathbb{C}^n$ with the usual addition and scalar multiplication
4. The set of $m \times n$ matrices with real entries $\mathbb{R}^{m\times n}$ with matrix addition and scalar multiplication
5. The set of positive real numbers $\mathbb{R}^+$ with the addition operation by

   $$u \oplus v = uv \quad \text{for all } u \text{ and } v \text{ in } \mathbb{R}^+$$

   and scalar multiplication operation by

   $$c \circ u = u^c \quad \text{for all } u \text{ in } \mathbb{R}^+ \text{ and scalar } c.$$

   Notice that the zero vector $\mathbf{0}$ in $\mathbb{R}^+$ is 1 and $(-u)=\dfrac{1}{u}$.

6. For $n \geq 0$, denote by $\mathbb{P}_n$ the set of polynomials of degree **at most** n, i.e., all polynomials of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

with coefficients $a_0, \ldots, a_n$ and variable x. For any polynomials p and q in $\mathbb{P}_n$,

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \text{and} \quad q(x) = b_0 + b_1 x + \cdots + b_n x^n,$$

the addition and scalar multiplication are defined as follows,

$$\begin{aligned} (p+q)(x) &= (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n \\ (cp)(x) &= ca_0 + (ca_1)x + \cdots + (ca_n)x^n. \end{aligned}$$

Then $\mathbb{P}_n$ is a vector space with $\mathbf{0} = 0(x) = 0$.

7. Let $C[a,b]$ be the set of all real-valued continuous functions defined on $[a,b]$ with the usual addition and scalar multiplication. i.e.,

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = c(f(x)).$$

Then $C[a,b]$ is a vector space.

Subspace

A subset S of a vector space V is called a subspace if S satisfies the following two properties.

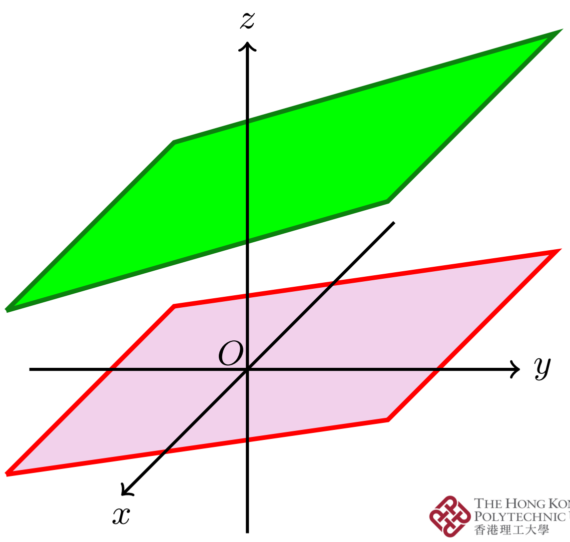
1. S is closed under addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in S, the sum $\mathbf{u}+\mathbf{v}$ is in S.
2. S is closed under scalar multiplication. That is, for each $\mathbf{u}$ in S and scalar c, the scalar $c\mathbf{u}$ is in S.

Subspace vs Vector space

Every subspace itself is a vector space.

Zero vector in subspace

If S is a subspace of a vector space V, then S must contain the zero vector $\mathbf{0}$ of V.



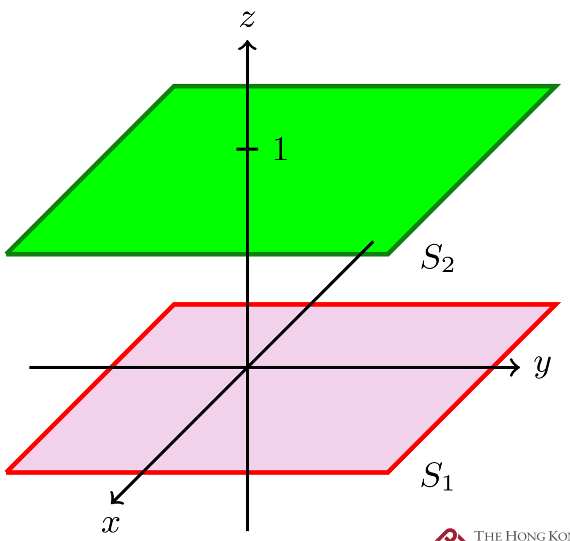
Example 1.5 Are $S_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$ and $S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$ subspaces of $\mathbb{R}^3$?

Proof. For any $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$ in S_1 and scalar c,

1. the sum $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ 0 \end{bmatrix}$ is in S_1.
2. the scalar multiple $c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ 0 \end{bmatrix}$ is also in S_1.

So S_1 is a subspace of $\mathbb{R}^3$.

Because the zero vector $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is NOT in S_2. S_2 is not a subspace.



Examples of Subspace

1. The set of all $n \times n$ upper triangular matrices T_n is a subspace of $\mathbb{R}^{n \times n}$.
2. The set of all $n \times n$ diagonal matrices D_n is a subspace of $\mathbb{R}^{n \times n}$.
3. The set of polynomials of degree at most n, $\mathbb{P}_n$, is a subspace the set of all real-valued continuous functions $C[a, b]$.
4. The set $S = \left\{ A = [a_{ij}] \in \mathbb{R}^{3\times 3} : a_{13} = 2a_{31} \right\}$ is a subspace of $\mathbb{R}^{3\times 3}$.

Proof of Example 4. For any $A = [a_{ij}]$ and $B = [b_{ij}]$ in S and scalar c,

1. the $(1, 3)$th entry of $A + B$ is
$$(a_{13} + b_{13}) = 2(a_{31} + b_{31}),$$
which is equal to two times of the $(3, 1)$th entry of $A + B$.
2. the $(1, 3)$th entry of cA is
$$ca_{13} = 2(ca_{31}),$$
which is the same as two times of the $(3, 1)$th entry of cA.

Thus, S is a subspace of $\mathbb{R}^{3\times 3}$.

Span $\Longrightarrow$ Subspace

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in V, then the set $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof. For any $\mathbf{u}$ and $\mathbf{v}$ in $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ and scalar α,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p \quad \text{and} \quad \mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_p\mathbf{v}_p$$

for some $c_1, \ldots, c_p$ and $d_1, \ldots, d_p$. Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \cdots + (c_p + d_p)\mathbf{v}_p$$

and

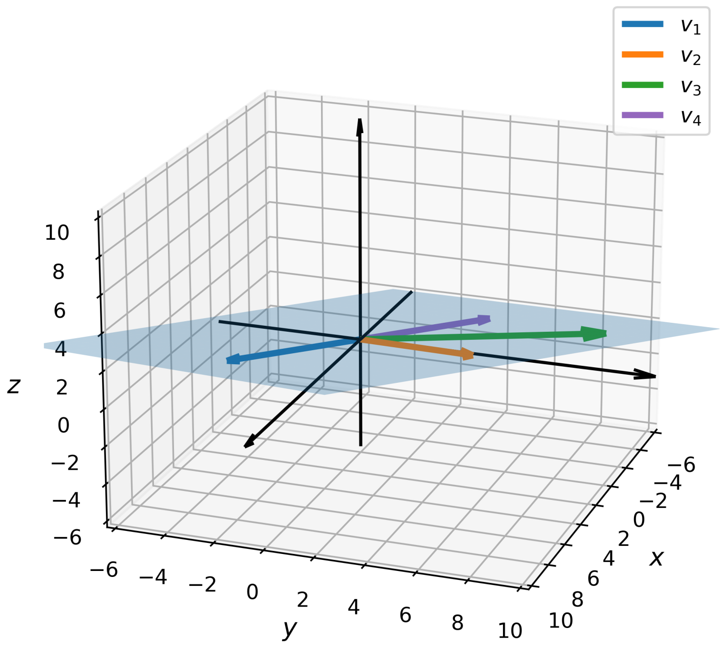
$$\alpha\mathbf{u} = (\alpha c_1)\mathbf{v}_1 + (\alpha c_2)\mathbf{v}_2 + \cdots + (\alpha c_p)\mathbf{v}_p.$$

Thus, $\mathbf{u} + \mathbf{v}$ and $\alpha\mathbf{u}$ are in $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$.

Example Let

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 10 \\ 2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_5 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}.$$

$$\begin{aligned}
& \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \\
= \; & \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \\
= \; & \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\} \\
= \; & \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\} \\
= \; & \operatorname{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \\
= \; & \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \\
= \; & \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_3\} \\
= \; & \operatorname{Span}\{\mathbf{v}_2, \mathbf{v}_3\} \\
= \; & \operatorname{Span}\{\mathbf{v}_2, \mathbf{v}_4\} \\
= \; & \operatorname{Span}\{\mathbf{v}_3, \mathbf{v}_4\} \\
\neq \; & \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_4\}
\end{aligned}$$



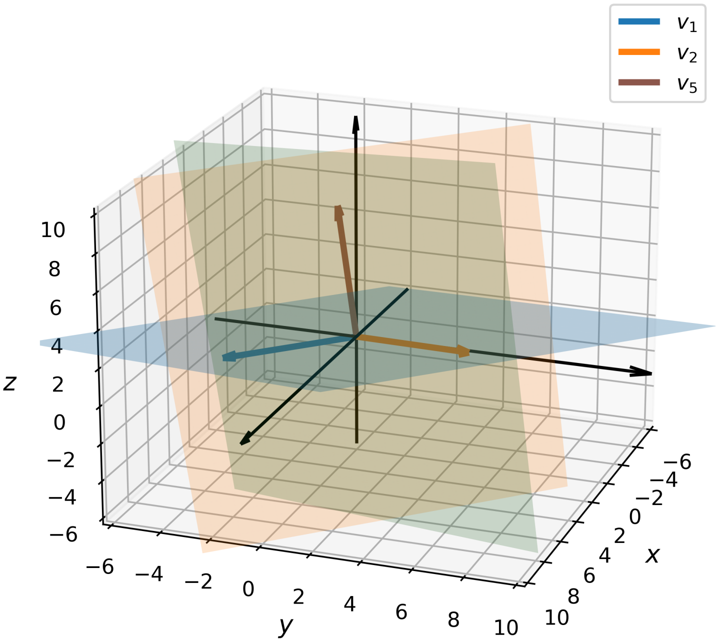
Example Let

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 10 \\ 2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_5 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}.$$

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq \text{Span}\{\mathbf{v}_1, \mathbf{v}_5\}$$
$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq \text{Span}\{\mathbf{v}_2, \mathbf{v}_5\}$$
$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_5\} \neq \text{Span}\{\mathbf{v}_2, \mathbf{v}_5\}$$

All the three subspaces of $\mathbb{R}^3$ are not the same. But

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\} = \mathbb{R}^3.$$



Example 1.6 Show that $S = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

Proof. For any $\mathbf{u}$ in S,

$$\mathbf{u} = \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ b \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, we see that

$$S = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Thus, S is subspace of $\mathbb{R}^4$.

Vector space and subspace

Example 1.7 Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ -4 \\ 5 \\ -2 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 4 \\ 8 \\ -8 \\ 9 \\ 4 \end{bmatrix} \text{ and } \mathbf{v}_5 = \begin{bmatrix} 4 \\ 8 \\ -8 \\ 10 \\ 4 \end{bmatrix},$$

and $S = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\, \mathbf{v}_4, \mathbf{v}_5\}$ is a subspace of $\mathbb{R}^5$.

1. Suppose $S = \text{Span}\,\{\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k, \mathbf{v}_\ell\}$ for some i, j, k, ℓ. What are these four vectors?
2. Can S be spanned by any three vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$? Why or why not?

Solution. Let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_5 \end{bmatrix}$. It follows from the RREF of A that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5$ are linearly independent and $\mathbf{v}_4$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5$. Thus,

$$S = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\}.$$

Furthermore, any of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5$ cannot be written a linear combination of the other three vectors, S cannot be spanned by three vectors.

[7]:
```
A = sp.Matrix([[1,-2,-5,3,1],␣
↪[2,5,6,-1,2],␣
↪[3,-4,5,-2,3],␣
↪[4,8,-8,9,4],␣
↪[4,8,-8,10,4]]).T; A
```

[7]:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ -2 & 5 & -4 & 8 & 8 \\ -5 & 6 & 5 & -8 & -8 \\ 3 & -1 & -2 & 9 & 10 \\ 1 & 2 & 3 & 4 & 4 \end{bmatrix}$$

[8]:
```
A.rref()
```

[8]:
```
(Matrix([
 [1, 0, 0,  3, 0],
 [0, 1, 0,  2, 0],
 [0, 0, 1, -1, 0],
 [0, 0, 0,  0, 1],
 [0, 0, 0,  0, 0]]),
 (0, 1, 2, 4))
```


Null space and column space

Column Space

The column space of an $m \times n$ matrix A, written as $\text{Col}\,A$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\text{Col}\,A = \text{Span}\,\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \{\mathbf{b} \in \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\},$$

which is subspace of $\mathbb{R}^m$.

Null Space

The null space of an $m \times n$ matrix A, written as $\text{Nul}\,A$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. i.e.,

$$\text{Nul}\,A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

The null space $\text{Nul}\,A$ is a subspace of $\mathbb{R}^n$.

Proof. Take any $\mathbf{u}$ and $\mathbf{v}$ in $\text{Nul}\,A$. Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Now

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Therefore, $\mathbf{u} + \mathbf{v}$ is in $\text{Nul}\,A$. For any $c \in \mathbb{R}$,

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c \cdot \mathbf{0} = \mathbf{0}.$$

Thus, $c\mathbf{u}$ is also in $\text{Nul}\,A$.

Null space and column space

Example 1.8 Let

$$A = \begin{bmatrix} 0 & -1 & -1 & 1 & -2 & 0 & 6 \\ -1 & -1 & 1 & 1 & -2 & -2 & 2 \\ 1 & 0 & -2 & 0 & 2 & -2 & -8 \\ 0 & 0 & 0 & 0 & 1 & -2 & -6 \\ 2 & 1 & -3 & -1 & 2 & -2 & -10 \\ -2 & -2 & 2 & 2 & 1 & 1 & 4 \\ 0 & -2 & -2 & 2 & 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -7 \\ -6 \\ 6 \\ 7 \\ 8 \\ 0 \\ 0 \end{bmatrix}.$$

1. Determine if $\mathbf{b}$ is in $\operatorname{Nul} A$.
2. Determine if $\mathbf{b}$ is in $\operatorname{Col} A$.

[1]:
```
A = sp.Matrix([[0,-1,-1,1,-2,0,6],␣
↪[-1,-1,1,1,-2,-2,2],␣
↪[1,0,-2,0,2,-2,-8], [0,0,0,0,1,-2,-6],␣
↪[2,1,-3,-1,2,-2,-10],␣
↪[-2,-2,2,2,1,1,4],␣
↪[0,-2,-2,2,1,-1,0]]); A
```

[1]:
$$\begin{bmatrix} 0 & -1 & -1 & 1 & -2 & 0 & 6 \\ -1 & -1 & 1 & 1 & -2 & -2 & 2 \\ 1 & 0 & -2 & 0 & 2 & -2 & -8 \\ 0 & 0 & 0 & 0 & 1 & -2 & -6 \\ 2 & 1 & -3 & -1 & 2 & -2 & -10 \\ -2 & -2 & 2 & 2 & 1 & 1 & 4 \\ 0 & -2 & -2 & 2 & 1 & -1 & 0 \end{bmatrix}$$

[2]:
```
b = sp.Matrix([[-7,-6,6,7,8,0,0]]).T; b
```

[2]:
$$\begin{bmatrix} -7 \\ -6 \\ 6 \\ 7 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

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Null space and column space

[3]:
```
A@b
```

[3]:
$$\begin{bmatrix} -9 \\ 10 \\ -3 \\ 8 \\ -29 \\ 60 \\ 22 \end{bmatrix}$$

[4]:
```
A.row_join(b)
```

[4]:
$$\begin{bmatrix} 0 & -1 & -1 & 1 & -2 & 0 & 6 & -7 \\ -1 & -1 & 1 & 1 & -2 & -2 & 2 & -6 \\ 1 & 0 & -2 & 0 & 2 & -2 & -8 & 6 \\ 0 & 0 & 0 & 0 & 1 & -2 & -6 & 7 \\ 2 & 1 & -3 & -1 & 2 & -2 & -10 & 8 \\ -2 & -2 & 2 & 2 & 1 & 1 & 4 & 0 \\ 0 & -2 & -2 & 2 & 1 & -1 & 0 & 0 \end{bmatrix}$$

[5]:
```
A.row_join(b).rref()
```

[5]:
```
(Matrix([
 [1, 0, -2,  0, 0, 0,  0, 0],
 [0, 1,  1, -1, 0, 0, -2, 0],
 [0, 0,  0,  0, 1, 0, -2, 0],
 [0, 0,  0,  0, 0, 1,  2, 0],
 [0, 0,  0,  0, 0, 0,  0, 1],
 [0, 0,  0,  0, 0, 0,  0, 0],
 [0, 0,  0,  0, 0, 0,  0, 0]]),
 (0, 1, 4, 5, 7))
```

Solution.

1. Compute the product $A\mathbf{b}$,

$$A\mathbf{b} = \begin{bmatrix} -9 \\ 10 \\ -3 \\ 8 \\ -29 \\ 60 \\ 22 \end{bmatrix} \neq \mathbf{0}.$$

   Obviously, $\mathbf{b}$ is not a solution of $A\mathbf{b} = \mathbf{0}$, so $\mathbf{b}$ is not in $\operatorname{Nul} A$.

2. We need to solve $A\mathbf{x} = \mathbf{b}$.

$$[A \mid \mathbf{b}]$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   We see that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, so $\mathbf{b}$ is not in $\operatorname{Col} A$.

Example 1.9 Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 2 & 0 & -3 & -3 & 4 \\ 0 & 1 & 1 & 1 & 1 & 2 & 0 & -3 & -8 & 1 \\ -2 & 0 & 2 & 2 & 2 & -2 & -2 & 6 & -2 & -6 \\ -1 & 2 & 3 & 3 & 0 & 1 & 2 & -4 & -7 & -2 \\ 2 & -1 & -3 & -3 & 0 & 2 & -1 & -2 & 0 & 6 \\ -2 & 2 & 4 & 4 & -2 & 2 & 4 & -6 & -6 & 2 \end{bmatrix}.$$

Method 1

[1]:
```
A = sp.Matrix([[1,0,-1,-1,0,2,0,-3,-3,4], [0,1,1,1,1,2,0,-3,-8,1],␣
↪[-2,0,2,2,2,-2,-2,6,-2,-6],[-1,2,3,3,0,1,2,-4,-7,-2], [2,-1,-3,-3,0,2,-1,-2,0,6],␣
↪[-2,2,4,4,-2,2,4,-6,-6,2]]); A
```

[1]:
$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 2 & 0 & -3 & -3 & 4 \\ 0 & 1 & 1 & 1 & 1 & 2 & 0 & -3 & -8 & 1 \\ -2 & 0 & 2 & 2 & 2 & -2 & -2 & 6 & -2 & -6 \\ -1 & 2 & 3 & 3 & 0 & 1 & 2 & -4 & -7 & -2 \\ 2 & -1 & -3 & -3 & 0 & 2 & -1 & -2 & 0 & 6 \\ -2 & 2 & 4 & 4 & -2 & 2 & 4 & -6 & -6 & 2 \end{bmatrix}$$

[2]:
```
x1,x2,x3,x4,x5,x6,x7,x8,x9,x10 = sp.symbols('x1 x2 x3 x4 x5 x6 x7 x8 x9 x10')
sp.linsolve((A,sp.zeros(10,1)),x1,x2,x3,x4,x5,x6,x7,x8,x9,x10)
```

[2]: $\{(x_3 + x_4 + x_8 - x_9,\ 2x_{10} - x_3 - x_4 - x_7 + 2x_8 + 2x_9, x_3,\ x_4,\ x_{10} + x_7 - x_8 + 2x_9,\ -2x_{10} + x_8 + 2x_9,\ x_7,\ x_8,\ x_9,\ x_{10})\}$

The general solution is of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_3 + x_4 + x_8 - x_9 \\ -x_3 - x_4 - x_7 + 2x_8 + 2x_9 + 2x_{10} \\ x_3 \\ x_4 \\ x_7 - x_8 + 2x_9 + x_{10} \\ x_8 + 2x_9 - 2x_{10} \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix}$$

$$= x_3 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_2} + x_7 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_3} + x_8 \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_4} + x_9 \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 2 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}_5} + x_{10} \underbrace{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{u}_6}$$

$$= x_3\mathbf{u}_1 + x_4\mathbf{u}_2 + x_7\mathbf{u}_3 + x_8\mathbf{v}_4 + x_9\mathbf{v}_5 + x_{10}\mathbf{v}_6.$$

So

$$\text{Nul}\, A = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$$

and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$ is the spanning set for $\text{Nul}\, A$.

Method 2

[3]:
```
A.nullspace()
```

[3]:
```
[Matrix([
 [ 1],
 [-1],
 [ 1],
 [ 0],
 [ 0],
 [ 0],
 [ 0],
 [ 0],
 [ 0],
 [ 0]]),
 Matrix([
 [ 1],
 [-1],
 [ 0],
 [ 1],
 [ 0],
 [ 0],
 [ 0],
 [ 0],
 [ 0],
 [ 0]]),
 Matrix([
 [ 0],
 [-1],
 [ 0],
 [ 0],
 [ 1],
 [ 0],
 [ 1],
 [ 0],
 [ 0],
 [ 0]]),
 Matrix([
 [ 1],
 [ 2],
 [ 0],
 [ 0],
 [-1],
 [ 1],
 [ 0],
 [ 1],
 [ 0],
 [ 0]]),
 Matrix([
 [-1],
 [ 2],
 [ 0],
 [ 0],
 [ 2],
 [ 2],
 [ 0],
 [ 0],
 [ 1],
 [ 0]]),
 Matrix([
 [ 0],
 [ 2],
 [ 0],
 [ 0],
 [ 1],
 [-2],
 [ 0],
 [ 0],
 [ 0],
 [ 1]])]
```

[4]:
```
# Rank of A @ SymPy
A.rank()
```

[4]: 4

[5]:
```
import numpy as np
```

[5]:
```
# Convert Matrix @ SymPy to NumPy Array
B = np.array(A).astype(np.float64)
```

[6]:
```
# Rank of B @ NumPy
np.linalg.matrix_rank(B)
```

[6]: 4

Rank

The rank of A, denoted by $\text{Rank}\,A$, is the number of pivot entries of A.

Let

$$\mathbf{u}_1 = \begin{bmatrix}3\\0\\0\end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix}0\\2\\0\end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix}0\\0\\3\end{bmatrix}, \quad \text{and} \quad \mathbf{u}_4 = \begin{bmatrix}3\\-1\\1\end{bmatrix}.$$

Clearly,

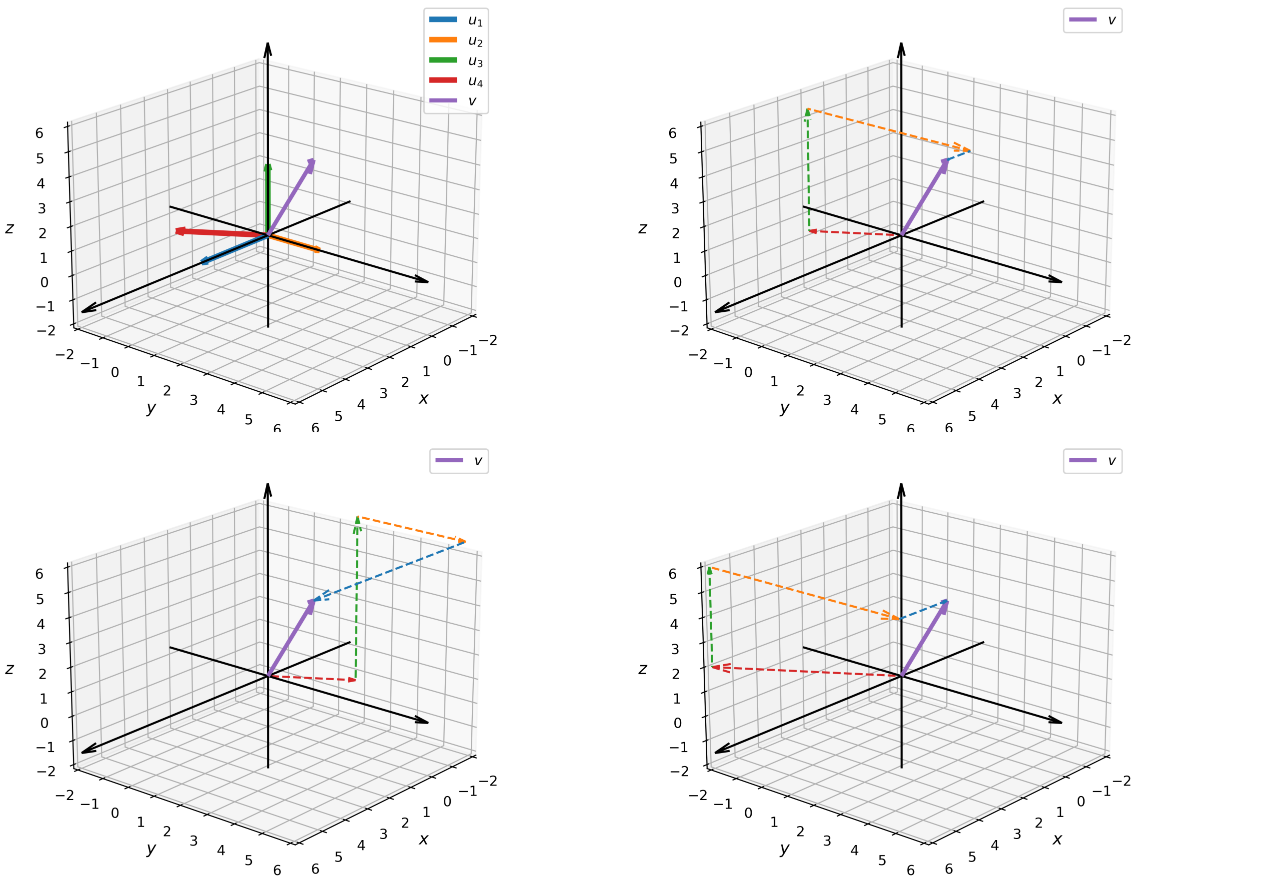
$$\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \mathbb{R}^3$$

as every vector in the vector space $\mathbb{R}^3$ are be written as linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$. But..... if $\mathbf{v} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, then

$$\begin{aligned}
\mathbf{v} &= \frac{1}{3}\mathbf{u}_1 + 3\mathbf{u}_2 + \frac{5}{3}\mathbf{u}_3 + \mathbf{u}_4, \\
\mathbf{v} &= \frac{7}{3}\mathbf{u}_1 + 2\mathbf{u}_2 + \frac{7}{3}\mathbf{u}_3 - \mathbf{u}_4, \\
\mathbf{v} &= -\frac{2}{3}\mathbf{u}_1 + \frac{7}{2}\mathbf{u}_2 + \frac{4}{3}\mathbf{u}_3 + 2\mathbf{u}_4
\end{aligned}$$

The combination may NOT be **unique**.

Basis and dimension



Basis and dimension

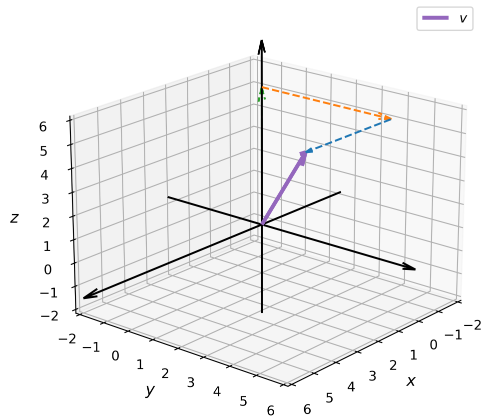
If we just consider

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

Again, $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbb{R}^3$. Now if $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, then

$$\mathbf{v} = \frac{4}{3}\mathbf{u}_1 + \frac{5}{2}\mathbf{u}_2 + 2\mathbf{u}_3.$$

The combination is now **unique**.



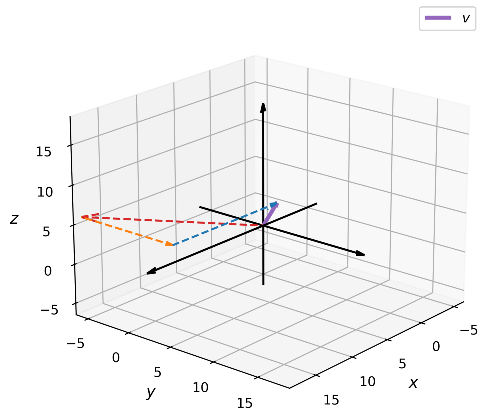
Even if we just consider

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

One can show that $\operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4\} = \mathbb{R}^3$ and

$$\mathbf{v} = -\frac{14}{3}\mathbf{u}_1 + \frac{11}{2}\mathbf{u}_2 + 6\mathbf{u}_4.$$

The combination is also **unique**.



[1]:
```
u1 = sp.Matrix([[3,0,0]]).T;
u2 = sp.Matrix([[0,2,0]]).T;
u3 = sp.Matrix([[0,0,3]]).T;
u4 = sp.Matrix([[3,-1,1]]).T;
v = sp.Matrix([[4,5,6]]).T;
```

[2]:
```
x1,x2,x3,x4 = sp.symbols('x1 x2 x3 x4')
```

[3]:
```
# A = [u1 u2 u3 u4]
A = sp.BlockMatrix([u1,u2,u3,u4]).
↪as_explicit(); A
```

[3]: $\begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

[4]:
```
v
```

[4]: $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

[5]:
```
sp.linsolve((A,v),x1,x2,x3,x4)
```

[5]: $\left\{\left(\frac{4}{3} - x_4,\ \frac{x_4}{2} + \frac{5}{2},\ 2 - \frac{x_4}{3},\ x_4\right)\right\}$

[6]:
```
# A = [u1 u2 u3]
A = sp.BlockMatrix([u1,u2,u3]).
↪as_explicit(); A
```

[6]: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

[7]:
```
sp.linsolve((A,v),x1,x2,x3)
```

[7]: $\left\{\left(\frac{4}{3},\ \frac{5}{2},\ 2\right)\right\}$

[8]:
```
# A = [u1 u2 u4]
A = sp.BlockMatrix([u1,u2,u4]).
↪as_explicit(); A
```

[8]: $\begin{bmatrix} 3 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

[9]:
```
sp.linsolve((A,v),x1,x2,x3)
```

[9]: $\left\{\left(-\frac{14}{3},\ \frac{11}{2},\ 6\right)\right\}$

Basis for a vector space

An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in a vector space V is a **basis** for V if

1. $\mathcal{B}$ is a linearly independent set, and
2. the subspace spanned by $\mathcal{B}$ coincides with V, that is,

$$\operatorname{Span}\mathcal{B} = \operatorname{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\} = V.$$

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Dimension of a Vector Space

- The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.
- If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as $\dim V$, is the number of vectors in a basis for V.
- If V is not spanned by a finite set, then V is said to be infinite-dimensional.

Proof. Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V. Then $V = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. So for any vector $\mathbf{x}$ in V, $\mathbf{x}$ can be written as a linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_n$.

Suppose a vector $\mathbf{x}$ in V has two representations,

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \quad \text{and} \quad \mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n,$$

for some scalars $c_1, \ldots, c_n$ and $d_1, \ldots, d_n$. Then

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_n - d_n)\mathbf{b}_n.$$

As $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is linearly independent, we must have

$$(c_1 - d_1) = 0 \quad (c_2 - d_2) = 0 \quad \cdots \quad (c_n - d_n) = 0.$$

i.e.,

$$c_1 = d_1 \quad c_2 = d_2 \quad \cdots \quad c_n = d_n.$$

Therefore, the representation is unique.

Basis for a subspace

An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ in a subspace S of a vector space V is a **basis** for S if

1. $\mathcal{B}$ is a linearly independent set, and
2. the subspace spanned by $\mathcal{B}$ coincides with S, that is,

$$\text{Span}\,\mathcal{B} = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_k\} = S.$$

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for a subspace S. Then for each $\mathbf{x}$ in S, there exists a unique set of scalars $c_1, \ldots, c_k$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k.$$

Dimension of a subspace

Let S be a subspace of a finite-dimensional vector space V. Then S is finite-dimensional and

$$\dim S \leq \dim V.$$

Proof. Will be provided later.

Examples

1. The sets $\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$ are two bases for $\mathbb{R}^3$.
2. Let
$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$
The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis for $\mathbb{R}^4$. This basis is called the **standard basis** for $\mathbb{R}^4$.
3. In general, let $\mathbf{e}_j$ be the vector in $\mathbb{R}^n$ with 1 in the jth entry and zero elsewhere. Then
   - $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is linearly independent, and
   - every vector can be written as a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_n$.

   Thus, $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$. This basis is called the **standard basis** for $\mathbb{R}^n$.

4. Let
$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
The set $\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ forms a basis for $\mathbb{R}^{2\times 3}$.
5. In general let E_{ij} be the $m \times n$ with 1 in the (i, j)th entry and zero elsewhere. Then
$$\{E_{11}, E_{12}, \dots, E_{1n}, \dots \cdots, E_{m1}, \dots, E_{mn}\}$$
forms a basis for $\mathbb{R}^{m\times n}$. This is called the **standard basis** for $\mathbb{R}^{m\times n}$.
6. Let
$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2, \quad \dots \quad p_n(x) = x^n.$$
The set $\{p_0, p_1, p_2, \dots, p_n\}$ forms a basis for $\mathbb{P}_n$, the space of polynomials of degree at most n.

Example Let

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2, \quad \dots \quad p_n(x) = x^n$$

and $\mathcal{B} = \{p_0, p_1, p_2, \dots, p_n\}$.

- For any scalars $c_0, \dots, c_n$,

$$c_0 p_0(x) + c_1 p_1(x) + \dots + c_n p_n(x) = \mathbf{0} \quad \Longrightarrow \quad c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + \dots + c_n \cdot x^n = \mathbf{0}.$$

  Then we must have $c_0 = c_1 = c_2 = \dots = c_n = 0$. Therefore, the polynomial $p_0, p_1, p_2, \dots, p_n$ are linearly independent.

- For any polynomial $p(x)$ in $\mathbb{P}_n$, $p(x)$ must have the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = a_0 p_0(x) + a_1 p_1(x) + a_2 p_2(x) + \dots + a_n p_n(x).$$

  for some coefficients $a_0, a_1, a_2, \dots, a_n$. Thus, $p(x)$ is a linear combination of $p_0, p_1, p_2, \dots, p_n$, or equivalently, $p(x)$ is in $\text{Span}\,\{p_0, p_1, p_2, \dots, p_n\}$.

In summary , $\mathcal{B} = \{p_0, p_1, p_2, \dots, p_n\}$ forms a basis for $\mathbb{P}_n$.

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Example 1.9 (cont.) Find a basis for $\text{Nul}\,A$ and $\text{Col}\,A$ respectively, where

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 2 & 0 & -3 & -3 & 4 \\ 0 & 1 & 1 & 1 & 1 & 2 & 0 & -3 & -8 & 1 \\ -2 & 0 & 2 & 2 & 2 & -2 & -2 & 6 & -2 & -6 \\ -1 & 2 & 3 & 3 & 0 & 1 & 2 & -4 & -7 & -2 \\ 2 & -1 & -3 & -3 & 0 & 2 & -1 & -2 & 0 & 6 \\ -2 & 2 & 4 & 4 & -2 & 2 & 4 & -6 & -6 & 2 \end{bmatrix}.$$

Solution. Reduce A to its reduced row echelon form

$$A \sim \dots \sim \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution of $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = x_3 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_1} + x_4 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_2} + x_7 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_3} + x_8 \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{u}_4} + x_9 \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 2 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{u}_5} + x_{10} \underbrace{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{u}_6}$$

So $\text{Nul}\,A = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$. These six vectors are linearly independent (why?). Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$ forms a basis for $\text{Nul}\,A$. Also $\dim \text{Nul}\,A = 6$.

From the reduced row echelon form of A,

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ -1 \\ 2 \end{bmatrix} \quad \mathbf{a}_5 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \\ -2 \end{bmatrix} \quad \mathbf{a}_6 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

are linearly independent and span the column space of A. Therefore, the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5, \mathbf{a}_6\}$ forms a basis for $\operatorname{Col} A$. Thus, $\dim \operatorname{Col} A = 4$.

[4]:
```
# Column Space of A
A.columnspace()
```

[4]:
```
[Matrix([
 [ 1],
 [ 0],
 [-2],
 [-1],
 [ 2],
 [-2]]),
 Matrix([
 [ 0],
 [ 1],
 [ 0],
 [ 2],
 [-1],
 [ 2]]),
 Matrix([
 [ 0],
 [ 1],
 [ 2],
 [ 0],
 [ 0],
 [-2]]),
 Matrix([
 [ 2],
 [ 2],
 [-2],
 [ 1],
 [ 2],
 [ 2]])]
```


Basis for null space

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ produced in parametric vector form of general solution of $A\mathbf{x} = \mathbf{0}$ are **always** linearly independent and from a basis for $\operatorname{Nul} A$.

Basis for column space

The pivot columns of a matrix A form a basis for $\operatorname{Col} A$.

Outline of proof.

- Every non-pivot column of A is a linear combination of pivot columns of A.
- The pivot columns of A are linearly independent.

Thus, the pivot columns of A form a basis of $\operatorname{Col} A$.

The Spanning Set Theorem

Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V, and let $S = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} \neq \{\mathbf{0}\}$.

1. If one of the vectors in B, say $\mathbf{v}_k$ is a linear combination of the remaining vectors in B, then the set formed from B by removing $\mathbf{v}_k$ still spans S, i.e.,
$$S = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_p\}.$$
2. Some subset of B forms a basis for S.

Proof of (1.) Clearly, $\mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_p\} \subseteq S$.
Suppose $\mathbf{v}_k$ is a linear combination of remaining vectors in B. Then
$$\mathbf{v}_k = d_1\mathbf{v}_1 + \cdots + d_{k-1}\mathbf{v}_{k-1} + d_{k+1}\mathbf{v}_{k+1} + \cdots + d_p\mathbf{v}_p.$$
Now for any $\mathbf{b}$ in $S = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$,
$$\begin{aligned}\mathbf{b} &= c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1} + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} + \cdots + c_p\mathbf{v}_p \\ &= (c_1 + c_kd_1)\mathbf{v}_1 + \cdots + (c_{k-1} + c_kd_{k-1})\mathbf{v}_{k-1} \\ &\quad +(c_{k+1} + c_kd_{k+1})\mathbf{v}_{k+1} + \cdots + (c_p + c_kd_p)\mathbf{v}_p,\end{aligned}$$
so $\mathbf{b}$ is also $\mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_p\}$. Thus,
$$S \subseteq \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_p\} \implies S = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \dots, \mathbf{v}_p\}.$$

Basis vs Invertible matrix

Let A be an invertible $n \times n$ matrix. Then the columns of A form a basis of $\mathbb{R}^n$.

Proof. Suppose A is invertible. Then

A is invertible.

$\Rightarrow$ A has a pivot entry in every column.

$\Rightarrow$ The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has trivial solution only.

$\Rightarrow$ The column vectors of A are linearly independent.

On the other hand,

A is invertible.

$\Rightarrow$ A has a pivot entry in every column.

$\Rightarrow$ The linear equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^n$.

$\Rightarrow$ The column vectors of A span $\mathbb{R}^n$.

Thus, the column vectors of A form a basis of $\mathbb{R}^n$

Example 1.10 Let

$$A = \begin{bmatrix} 2 & 8 & 6 & 0 & -6 & 8 & 0 \\ -1 & 8 & -2 & 1 & 1 & -9 & 5 \\ -5 & 7 & 3 & -1 & 7 & 2 & 7 \\ 3 & -5 & -1 & -6 & 7 & 3 & -4 \\ -5 & 2 & 2 & 6 & -9 & -7 & 3 \\ -8 & -8 & -5 & 8 & -5 & 8 & -7 \\ 8 & -9 & -7 & 8 & 6 & 1 & 2 \end{bmatrix}.$$

Determine if the columns of A form a basis for $\mathbb{R}^7$.

Three different solutions:

- Since the reduced row echolon form of A is I_7, the columns of A are linearly independent and hence they form a basis for $\mathbb{R}^7$.
- Since Rank $A = 7$, the columns of A are linearly independent and hence they form a basis for $\mathbb{R}^7$.
- Since $\det A = -2679572$, the matrix A is invertible and hence the columns of A form a basis for $\mathbb{R}^7$.

[8]:
```
A = sp.Matrix([[2,8,6,0,-6,8,0],␣
↪[-1,8,-2,1,1,-9,5],␣
↪[-5,7,3,-1,7,2,7],␣
↪[3,-5,-1,-6,7,3,-4],␣
↪[-5,2,2,6,-9,-7,3],␣
↪[-8,-8,-5,8,-5,8,-7],␣
↪[8,-9,-7,8,6,1,2]]); A
```

[8]:
$$\begin{bmatrix} 2 & 8 & 6 & 0 & -6 & 8 & 0 \\ -1 & 8 & -2 & 1 & 1 & -9 & 5 \\ -5 & 7 & 3 & -1 & 7 & 2 & 7 \\ 3 & -5 & -1 & -6 & 7 & 3 & -4 \\ -5 & 2 & 2 & 6 & -9 & -7 & 3 \\ -8 & -8 & -5 & 8 & -5 & 8 & -7 \\ 8 & -9 & -7 & 8 & 6 & 1 & 2 \end{bmatrix}$$

[9]:
```
A.rref()
```

[9]:
```
(Matrix([
 [1, 0, 0, 0, 0, 0, 0],
 [0, 1, 0, 0, 0, 0, 0],
 [0, 0, 1, 0, 0, 0, 0],
 [0, 0, 0, 1, 0, 0, 0],
 [0, 0, 0, 0, 1, 0, 0],
 [0, 0, 0, 0, 0, 1, 0],
 [0, 0, 0, 0, 0, 0, 1]]),
 (0, 1, 2, 3, 4, 5, 6))
```

[10]:
```
A.rank()
```

[10]: 7

[11]:
```
A.det()
```

[11]: -2679572

Basis vs linearly dependent

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Proof. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be a set in V with $p > n$. Then

$$\mathbf{u}_j = \sum_{i=1}^{n} c_{ij}\mathbf{b}_i \quad j = 1, \dots, p.$$

Then

$$\mathbf{0} = \sum_{j=1}^{p} x_j\mathbf{u}_j = \sum_{j=1}^{p} x_j \left(\sum_{i=1}^{n} c_{ij}\mathbf{b}_i\right) = \sum_{j=1}^{p}\sum_{i=1}^{n} c_{ij}x_j\mathbf{b}_i = \sum_{i=1}^{n}\left(\sum_{j=1}^{p} c_{ij}x_j\right)\mathbf{b}_i.$$

Since $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is linearly independent,

$$\sum_{j=1}^{p} c_{ij}x_j = 0 \quad j = 1, \dots, n \quad\Longrightarrow\quad \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{np} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \mathbf{0}$$

Since $p > 0$, the homogeneous system has nontrivial solutions and thus, not all x_j are zero. Therefore, $\mathbf{u}_1, \dots, \mathbf{u}_p$ are linearly dependent.

Number of vectors in different bases

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Proof. Suppose

$$\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \quad \text{and} \quad \mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$$

are both basis of V.

- Since $\mathcal{B}$ has n vectors and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are linearly independent,
$$k \le n.$$
- Since $\mathcal{V}$ has k vectors and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are linearly independent,
$$n \le k.$$

Hence, $k = n$.

Dimension of Subspace

Let S be a subspace of a finite-dimensional vector space V. Then S is finite-dimensional and $\dim S \le \dim V$.

Proof. Suppose $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis of S, i.e., $k = \dim S$. Then $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are k linearly independent vectors in V. As any set containing more than $\dim V$ vectors must be linearly dependent, we must have $k \le \dim V$.

Examples

1. The dimension of $\mathbb{R}^n$ is n.
2. The dimension of $\mathbb{R}^{n\times n}$ is n^2.
3. The dimension of $\mathbb{P}_n$ is $n+1$.
4. The dimension of the subspace of $n \times n$ diagonal matrices is n.
5. The dimension of the subspace of $n \times n$ triangular matrices is $\frac{1}{2}n(n+1)$.
6. Let $A = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 2 & 0 & -3 & -3 & 4 \\ 0 & 1 & 1 & 1 & 1 & 2 & 0 & -3 & -8 & 1 \\ -2 & 0 & 2 & 2 & 2 & -2 & -2 & 6 & -2 & -6 \\ -1 & 2 & 3 & 3 & 0 & 1 & 2 & -4 & -7 & -2 \\ 2 & -1 & -3 & -3 & 0 & 2 & -1 & -2 & 0 & 6 \\ -2 & 2 & 4 & 4 & -2 & 2 & 4 & -6 & -6 & 2 \end{bmatrix}$. Then
$$\dim \operatorname{Nul} A = 6 \quad \text{and} \quad \dim \operatorname{Col} A = 4 = \operatorname{Rank} A.$$

The Basis Theorem

Let V be a n-dimensional vector space, $n \ge 1$.

1. Any linearly independent set in V can be expanded, if necessary, to a basis for V.
2. Any set of exactly n linearly independent vectors in V forms a basis for V.
3. Any set of exactly n vectors that spans V forms a basis for V.

Dimension of $\operatorname{Nul} A$ and $\operatorname{Col} A$

- The dimension of $\operatorname{Nul} A$ is equal to the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$ (number of non-pivot columns),
- The dimension of $\operatorname{Col} A$ is equal to the number of pivot columns of A, i.e.,
$$\dim \operatorname{Col} A = \operatorname{Rank} A.$$
- If A is an $m \times n$ matrix, then
$$\dim \operatorname{Nul} A + \dim \operatorname{Col} A = n.$$

Example Find the dimensions of the null space and column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution. Row reduce the matrix A to its row echelon form.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\dim \operatorname{Nul} A = 3$ and $\dim \operatorname{Col} A = 2 = \operatorname{Rank} A$.

Row space

The row space of an $m \times n$ matrix A, written as $\operatorname{Row} A$, is the set of all linear combinations of the rows of A. If $A = \begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m \end{bmatrix}$, then

$$\operatorname{Row} A = \operatorname{Span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}.$$

Example If $A = \begin{bmatrix} -2 & -5 & 8 \\ 1 & 3 & -5 \\ 3 & 11 & -19 \\ 1 & 7 & -13 \end{bmatrix}$, then

$$\operatorname{Row} A = \operatorname{Span}\left\{ \begin{bmatrix} -2 & -5 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -5 \end{bmatrix}, \begin{bmatrix} 3 & 11 & -19 \end{bmatrix}, \begin{bmatrix} 1 & 7 & -13 \end{bmatrix} \right\}.$$

Row Space vs Row Equivalence

1. If two matrices A and B are row equivalent, then their row spaces are the same.
2. If B is in row echelon form, the nonzero rows of B form a basis for the row space of B as well as for that of A.

Proof. Notice that B is obtained from A by row operations.

- All rows of B are linear combinations of the rows of A.
- Any linear combinations of the rows of B is a linear combinations of the rows of A.
- The row space of B is contained in the row space of A.

By a similar argument, the row space of A is contained in the row space of B. Thus, the row space of B is the same as the row space of A.

Suppose B is in row each echelon form.

- All nonzero rows of B span the row space of B.
- All nonzero rows of B are linearly independent.

All nonzero rows of B form a basis for the row space of B and hence the row space of A.

Example 1.11 Find bases for the row space, the column space, and null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}.$$

Solution. Reduce A to its reduced row echelon form

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- The vectors

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 & -5 \end{bmatrix}$$

form a basis for Row A.

- The vectors $\begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix}$ form a basis for Col A.

Row space

- Write the general solution in parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

The vectors $\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$ form a basis for $\mathrm{Nul}\, A$.

- Finally,

$$\dim \mathrm{Col}\, A = \dim \mathrm{Row}\, A = 3 \quad \text{and} \quad \dim \mathrm{Nul}\, A = 2.$$

Rank

Rank

- The rank of A, denoted by $\mathrm{Rank}\, A$, is the dimension of the column space of A.
- The rank of A is also equal to the number of pivot entries of A.

The Rank Theorem

- The dimensions of the column space and the row space of an $m \times n$ matrix A are equal, i.e.,

$$\mathrm{Rank}\, A = \dim \mathrm{Col}\, A = \dim RowA.$$

- The rank of A satisfies the equation

$$\mathrm{Rank}\, A + \dim \mathrm{Nul}\, A = n.$$

Proof.

- $\dim \mathrm{Row}\, A =$ the number of pivot columns in A
- $\dim \mathrm{Col}\, A =$ the number of pivot columns in A
- $\dim \mathrm{Nul}\, A =$ the number of non-pivot columns in A

Example 1.12

1. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A.
2. Could a 6×9 matrix has a two dimensional null space?

Solution.

1. Since A has 9 columns, then $\text{Rank}\, A + \dim \text{Nul}\, A = 9$. i.e.,
$$\text{Rank}\, A = 9 - \dim \text{Nul}\, A = 9 - 2 = 7.$$
2. No!! If $\dim \text{Nul}\, A = 2$, then
$$\text{Rank}\, A = 9 - \dim \text{Nul}\, A = 9 - 2 = 7.$$
But $\text{Col}\, A$ is a subspace of $\mathbb{R}^6$. So
$$\text{Rank}\, A = \dim \text{Col}\, A \leq \dim \mathbb{R}^6 = 6!!$$
Contradiction!

Equivalent conditions for invertible matrix

Let A be a square $n \times n$ matrix. Then the following statements are equivalent.

1. A is an invertible matrix.
2. A is row equivalent to the identity matrix I_n.
3. A has n pivot positions.
4. The system $A\mathbf{x} = \mathbf{0}$ has only trivial solution.
5. The system $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.
6. The columns of A form a linearly independent set.
7. The columns of A span $\mathbb{R}^n$.
8. The columns of A form a basis of $\mathbb{R}^n$.
9. $\text{Col}\, A = \mathbb{R}^n$.
10. $\dim \text{Col}\, A = n$.
11. $\text{Rank}\, A = n$.
12. $\dim \text{Nul}\, A = 0$.
13. $\text{Nul}\, A = \{\mathbf{0}\}$.

$\mathcal{B}$-coordinate vector

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for a vector space V. Given a vector $\mathbf{x}$ in V.

- The coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\mathbf{x}$) are the weights $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

- The coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$ is defined by

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n.$$

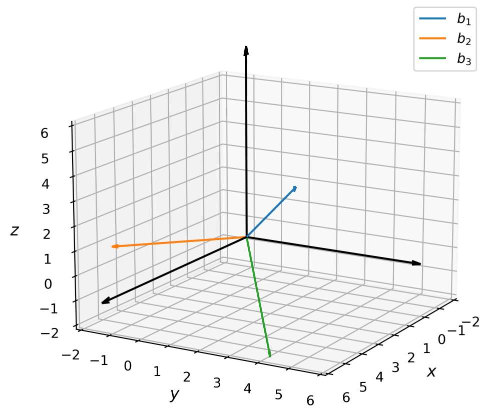
- The mapping

$$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$$

is called the coordinate mapping determined by $\mathcal{B}$.

Example Let

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}.$$



Then $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{R}^3$. Now

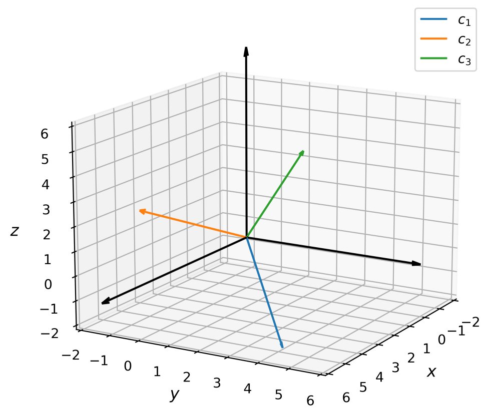
$$\mathbf{x} = 1 \cdot \mathbf{b}_1 + (-1) \cdot \mathbf{b}_2 + 2 \cdot \mathbf{b}_3$$

and so

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Now let

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}.$$



Then $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is another basis for $\mathbb{R}^3$. Here

$$\mathbf{x} = \frac{24}{7} \cdot \mathbf{c}_1 + \frac{-5}{7} \cdot \mathbf{c}_2 + \frac{2}{7} \cdot \mathbf{c}_3$$

and so

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} \frac{24}{7} \\ -\frac{5}{7} \\ \frac{2}{7} \end{bmatrix}.$$

Example Let $\mathcal{B} = \{1, x, x^2, x^3\}$ be the standard basis of the space $\mathbb{P}_3$ of polynomials of degree at most 3. For any $p(x)$ in $\in \mathbb{P}_3$, p has the form

$$p(t) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Then,

$$[p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^4.$$

Thus, the coordinate mapping $p \mapsto [p]_{\mathcal{B}}$ is an isomorphism form $\mathbb{P}_3$ onto $\mathbb{R}^4$. For example,

$$p_1(x) = 1 + 2x^3 \iff [p_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$p_2(x) = 4 + x + 5x^2 - x^3 \iff [p_2]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 5 \\ -1 \end{bmatrix}$$

Linear independence for coordinate vectors

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V. Then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in V are linearly independent if and only if the coordinate vectors $[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}$ in $\mathbb{R}^n$ are linearly independent.

Proof. The following statements are equivalent.

1. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in V are linearly dependent.
2. One of the vectors $\mathbf{v}_k$ is a linear combination of the remaining vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_p$.
3. One of the coordinate vectors $[\mathbf{v}_k]_{\mathcal{B}}$ is a linear combination of the remaining coordinate vectors $[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_{k-1}]_{\mathcal{B}}, [\mathbf{v}_{k+1}]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}$.
4. The coordinate vectors $[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_p]_{\mathcal{B}}$ in $\mathbb{R}^n$ are linearly dependent.

Example Let

$$\mathbf{x}_1 = \begin{bmatrix}1\\0\\2\\0\\0\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}0\\-1\\1\\1\\0\end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} \quad \mathbf{b} = \begin{bmatrix}4\\3\\12\\-3\\0\end{bmatrix}.$$

Show that $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ forms a basis for $S = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ and find $[\mathbf{p}]_\mathcal{B}$, the coordiniate vector of $\mathbf{b}$ relative to $\mathcal{B}$.

Solution. Let

$$A = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\0 & -1 & 0\\2 & 1 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}.$$

Since Rank $A = 3$, these three vectors are linearly independent. Thus, they form a basis for $S = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.
Now consider the linear system $A\mathbf{x} = \mathbf{b}$. The solution is

$$[\mathbf{b}]_\mathcal{B} = \mathbf{x} = \begin{bmatrix}11\\-3\\-7\end{bmatrix}.$$

[1]:
```
A = sp.Matrix([[1,0,2,0,0],
[0,-1,1,1,0], [1,0,1,0,0]]).T; A
```

[2]:
$$\begin{bmatrix}1 & 0 & 1\\0 & -1 & 0\\2 & 1 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

[2]:
```
b = sp.Matrix([[4,3,12,-3,0]]).T; b
```

[2]:
$$\begin{bmatrix}4\\3\\12\\-3\\0\end{bmatrix}$$

[3]:
```
x1,x2,x3= sp.symbols('x1 x2 x3')
```

[4]:
```
sp.linsolve((A,b),x1,x2,x3)
```

[4]: $\{(11, \ -3, \ -7)\}$

Change-of-coordinates matrix

Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2 \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$. Let

$$P_\mathcal{B} = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n\end{bmatrix}.$$

Then $P_\mathcal{B}$ is invertible.

For any vector $\mathbf{x}$ in $\mathbb{R}^n$ with coordinate vector $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix}$,

$$\mathbf{x} = P_\mathcal{B}[\mathbf{x}]_\mathcal{B} \iff [\mathbf{x}]_\mathcal{B} = P_\mathcal{B}^{-1}\mathbf{x}.$$

We call $P_\mathcal{B}$ the change-of-coordinates matrix form $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$. Furthermore, the coordinate mapping $x \mapsto [x]_\mathcal{B}$ can be defined by

$$\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B} = P_\mathcal{B}^{-1}\mathbf{x}.$$

Proof.

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}\begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix} = P_\mathcal{B}[\mathbf{x}]_\mathcal{B}.$$

Example 1.13 Use coordinate vectors to verify that polynomials

$$p_1(x) = 1 + 2x^2, \quad p_2(x) = 4 + x + 5x^2, \quad p_3(x) = 3 + 2x$$

are linearly dependent in $\mathbb{P}_2$.

Solution. Let $\mathcal{B} = \{1, x, x^2\}$ be the standard basis of $\mathbb{P}_2$. Then

$$[p_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad [p_2]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \quad [p_3]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

Since the matrix $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, the columns of A are linearly dependent, and hence

$$\{[p_1]_{\mathcal{B}}, [p_2]_{\mathcal{B}}, [p_3]_{\mathcal{B}}\} \text{ is linearly dependent.}$$

Thus, the corresponding polynomials are also linearly dependent. Indeed,

$$p_3(x) = 3 + 2x = -5(1 + 2x^2) + 2(4 + x + 5x^2) = -5p_1(x) + 2p_2(x).$$

Example 1.14 Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ with

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

and

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}.$$

Then

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{C}} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ -1 & 1 & 4 \end{bmatrix}.$$

Now if $\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$, then

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\mathbf{x} = \begin{bmatrix} \frac{24}{7} \\ -\frac{5}{7} \\ \frac{2}{7} \end{bmatrix}.$$

[1]:
```
PB = sp.Matrix([[1,1,1],
[1,-1,1], [1,0,-2]])
PC = sp.Matrix([[1,2,0],
[1,0,2], [-1,1,4]])
x = sp.Matrix([[2,4,-3]]).T
```

[2]:
```
PB.inv()@x
```

[2]: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

[3]:
```
PC.inv()@x
```

[3]: $\begin{bmatrix} \frac{24}{7} \\ -\frac{5}{7} \\ \frac{2}{7} \end{bmatrix}$

[4]:
```
PC.inv()*PB
```

[4]: $\begin{bmatrix} \frac{3}{7} & -\frac{3}{7} & \frac{9}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{1}{7} \\ \frac{2}{7} & -\frac{2}{7} & -\frac{1}{7} \end{bmatrix}$

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases for $\mathbb{R}^n$. For any vector $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \quad \text{and} \quad \mathbf{x} = P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}}.$$

Then

$$P_{\mathcal{C}}[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} \implies [\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

Let $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}$. Then

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

The matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is called change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Denote $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = P_{\mathcal{B}} \quad \text{and} \quad P_{\mathcal{B}\leftarrow\mathcal{E}} = P_{\mathcal{B}}^{-1}.$$

Furthermore,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}\leftarrow\mathcal{E}} P_{\mathcal{E}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}.$$

Example 1.14 (cont.) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ with

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}.$$

Then

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{C}} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ -1 & 1 & 4 \end{bmatrix}$$

and

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}} = \frac{1}{7} \begin{bmatrix} 3 & -3 & 9 \\ 2 & 5 & -1 \\ 2 & -2 & -1 \end{bmatrix}.$$

In fact,

$$[\mathbf{b}_1]_{\mathcal{C}} = \frac{1}{7} \begin{bmatrix} 7 \\ 2 \\ 2 \end{bmatrix}, \quad [\mathbf{b}_2]_{\mathcal{C}} = \frac{1}{7} \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix}, \quad [\mathbf{b}_3]_{\mathcal{C}} = \frac{1}{7} \begin{bmatrix} 9 \\ -1 \\ -1 \end{bmatrix}.$$