3724 cheatsheet

A first-order differential equation: $\frac{dy}{dx} = f(x,y)$

A solution of DE is a funtion $y(x)$ that satisfies $\frac{d(y(x))}{dt} = f(x, y(x))$

A IVP: $\frac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$

separable: $\frac{dy}{dx} = f(x,y) = g(y)h(x) \Rightarrow \frac{dy}{g(y)} = h(x)dx \Rightarrow \int \frac{1}{g(y)} dy = \int h(x) dx + C$

homogeneous: $\frac{dy}{dx} = f(x,y) = g(\frac{y}{x}) \Rightarrow u = \frac{y}{x} \quad y = ux \quad \frac{dy}{dx} = u + x\frac{du}{dx}$

$$u + x\frac{du}{dx} = g(u) \Rightarrow \int \frac{du}{g(u)-u} = \int \frac{1}{x} dx + C$$

Linear equation: $\frac{dy}{dx} + a(x)y = b(x) \qquad u(x) = e^{\int a(x)dx}$

$$u(x)\frac{dy}{dx} + u(x)a(x)y = u(x)b(x) \Rightarrow \frac{du(x)y}{dx} = u(x)b(x)$$

$$\Rightarrow y = \frac{1}{u(x)}\left(\int u(x)b(x)dx + C\right)$$

Exact equation: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\begin{cases} \frac{\partial f}{\partial x} = M(x,y) \\ \frac{\partial f}{\partial y} = N(x,y) \end{cases} \Rightarrow \begin{array}{l} f(x,y) = \int M(x,y)\,dx + g(y) \\ N(x,y) = \int \frac{\partial M(x,y)}{\partial y} dx + g(y)' \end{array}$$

$$\Rightarrow f(x,y) = \int M(x,y)dx + \int \left(N(x,y) - \int \frac{\partial M}{\partial y} dx\right) dy = C$$

<u>Existence Theorem</u>: if $f(x,y)$ is a continuous function in $[a,b]\times[c,d]$ if (x_0,y_0) is a point in the area then $\exists\, \varepsilon > 0$, $y(x)$ defined in $(x_0-\varepsilon, x_0+\varepsilon)$ satisfies IVP $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

<u>Uniqueness Theorem</u>: $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous then $y_1(x)$ $y_2(x)$ satisfies IVP and $y_1(x) = y_2(x)$

Second order ordinary DE: $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$

When $P(x)$ and $Q(x)$ are constant, the DE called DE with constant coefficients

When $R(x) = 0$, the DE is homogeneous

<u>Existence and uniqueness Theorem</u>: IVP $\Rightarrow \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$, $y(x_0) = y_0 \quad y'(x_0) = y_0'$

where $P(x)$, $Q(x)$ and $R(x)$ are continous in an open interval containing x_0

Then there exist an unique solution $y(x)$

The general solution of homogeneous equation: $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$

y_1 and y_2 are two solution, if Wronskian of y_1 and y_2 $W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2$

is non-zero for some $x=x_0$, then y_1 and y_2 are linearly independent

for $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$

Let a, b be the root of $\lambda^2+p\lambda+q=0$

① if a and b are real and distinct, $y_1=e^{ax}$ and $y_2=e^{bx}$ $y(x)=C_1e^{ax}+C_2e^{bx}$

② if a and b are real and equal, $y_1=e^{ax}$ and $y_2=xe^{ax}$ $y(x)=C_1e^{ax}+C_2xe^{ax}$

③ if $a=\alpha+i\beta$ and $b=\alpha-i\beta$, $y_1=e^{\alpha x}\cos\beta x$ and $y_2=e^{\alpha x}\sin\beta x$ $y(x)=C_1e^{\alpha x}\cos\beta x+C_2e^{\alpha x}\sin\beta x$

General solution - Non-homogeneus

for $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$

$y_p(x)$ is a particular solution, and y_1, y_2 are two linearly independent solutions of homogeneous

the general solution: $y(x)=C_1y_1(x)+C_2y_2(x)+y_p(x)$

$y_p(x)=V_1(x)y_1(x)+V_2(x)y_2(x)$ assume $\begin{cases} V_1'y_1+V_2'y_2=0 \\ V_1'y_1'+V_2'y_2'=R(x) \end{cases}$

$V_1'(x)=-\frac{y_2(x)R(x)}{W(y_1,y_2)}$ $V_2'(x)=\frac{y_1(x)R(x)}{W(y_1,y_2)}$

$V_1(x)=-\int\frac{y_2(x)R(x)}{W(y_1,y_2)}dx$ $V_2(x)=\int\frac{y_1(x)R(x)}{W(y_1,y_2)}dx$

$y_p(x)=\left(-\int\frac{y_2(x)R(x)}{W(y_1,y_2)}dx\right)y_1(x)+\left(\int\frac{y_1(x)R(x)}{W(y_1,y_2)}dx\right)y_2(x)$

for $y''+py'+qy=R(x)$

if $y_1(x)=e^{ax}$ and $y_2(x)=e^{bx}$ then $W(y_1,y_2)=\begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix}=(b-a)e^{(a+b)x}$

$y_1(x)=e^{ax}$ and $y_2(x)=xe^{ax}$ then $W(y_1,y_2)=\begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & (ax+1)e^{ax} \end{vmatrix}=e^{2ax}$

$y_1(x)=e^{\alpha x}\cos\beta x$ and $y_2(x)=e^{\alpha x}\sin\beta x$ then $W(y_1,y_2)=\begin{vmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x \\ e^{\alpha x}(\alpha\cos\beta x-\beta\sin\beta x) & e^{\alpha x}(\alpha\sin\beta x+\beta\cos\beta x) \end{vmatrix}=\beta e^{2\alpha x}$

Alternative method

① $R(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ $\quad y_p(x) = C_n x^n + C_{n-1}x^{n-1} + \cdots + C_1 x + C_0$

② $R(x) = e^{ax}$ $\quad y_p(x) = Ce^{ax}$

③ $R(x) = \cos bx \,/\, \sin bx$ $\quad y_p(x) = C_1 \cos bx + C_2 \sin bx$

④ $e^{ax}\cos bx \,/\, e^{ax}\sin bx$ $\quad y_p(x) = C_1 e^{ax}\cos bx + C_2 e^{ax}\sin bx$

System of homogeneous DEs $\quad X'(t) = AX(t)$

$$X(t) = C_1 X_1(t) + C_2 X_2(t) + \cdots + C_n X_n(t)$$

$Q_d = a - bP \quad Q_s = -c + dP \quad P^* = \frac{a+c}{b+d} \quad Q^* = \frac{ad-bc}{b+d}$

Solow Growth Model

$K \to$ 资本 $\quad L \to$ 劳动力 $\quad Y \to$ 资本劳动力产出 $\quad Y = f(K, L)$

① $f(\lambda K, \lambda L) = \lambda f(K, L)$ $\quad$ ② $\frac{dK}{dt} = sY$ $\quad s$ 为投资系数 $\quad$ ③ $\frac{dL}{dt} = \lambda L$ $\quad \lambda$ 为增长系数

Therefore $Y = f(K, L) = f(L \cdot \frac{K}{L}, L \cdot 1) = L f(\frac{K}{L}, 1) = L\phi(k)$ where $k = \frac{K}{L}$

$$\frac{dK}{dt} = sY = sL\phi(k) \quad \text{Also } K = kL \Rightarrow \frac{dK}{dt} = L\frac{dk}{dt} + k\frac{dL}{dt} \Rightarrow sL\phi(k) = L\frac{dk}{dt} + k\lambda L$$

$$\Rightarrow \frac{dk}{dt} = s\phi(k) - \lambda k$$

Partial Differential Equation

A second order linear partial DE: $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$

if $G = 0$, it's homogeneous

for $\frac{d^2 y}{dx^2} + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$

case 1 $\lambda = 0$ Then

$$\frac{d^2 y}{dx^2} = 0 \Rightarrow \frac{dy}{dx} = \beta \Rightarrow y = \beta x + \alpha$$

Next, $y(0) = 0$ and $y(L) = 0$ imply $\begin{cases} \alpha = 0 \\ \beta L + \alpha = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$

The solution is $y(x) = 0$

Case 2 $\lambda<0$

$\mu^2+\lambda=0 \Rightarrow \mu=\pm\sqrt{-\lambda} \Rightarrow y=\alpha e^{\sqrt{-\lambda}x}+\beta e^{-\sqrt{-\lambda}x}$

the same $\alpha=0$ $\beta=0$

Case 3 $\lambda>0$ $\mu^2+\lambda=0 \Rightarrow \mu=\pm\sqrt{\lambda}i \Rightarrow y=\alpha\cos\sqrt{\lambda}x+\beta\sin\sqrt{\lambda}x$

Next $y(0)=0$ and $y(L)=0$ imply $\begin{cases}\alpha=0\\ \alpha\cos\sqrt{\lambda}L+\beta\sin\sqrt{\lambda}L=0\end{cases} \Rightarrow \begin{cases}\alpha=0\\ \beta\sin\sqrt{\lambda}L=0\end{cases}$

Notice $\sin\sqrt{\lambda}L=0 \Leftrightarrow \sqrt{\lambda}L=n\pi \Leftrightarrow \sqrt{\lambda}=\frac{n\pi}{L}$

Here $\lambda=(\frac{n\pi}{L})^2$ called eigenvalue of the boundary value problem

and $y_n(x)=\sin\frac{n\pi x}{L}$ is eigenfunction

for $\frac{d^2y}{dx^2}+\lambda y=0$ with $y'(0)=0$ and $y'(L)=0$

Case 1 $\lambda=0 \Rightarrow$ $y(x)=\alpha$

Case 2 $\lambda<0 \Rightarrow y(x)=0$

Case 3 $\lambda>0$ $y=\alpha\cos\sqrt{\lambda}x+\beta\sin\sqrt{\lambda}x$ Next $\Rightarrow \begin{cases}\beta=0\\ \alpha\sin\sqrt{\lambda}L=0\end{cases}$

Thus $y(x)=\alpha\cos\frac{n\pi}{L}x$ $\lambda=(\frac{n\pi}{L})^2$

for $y''(x)+\lambda y(x)=0$ $0<x<L,$ $\begin{cases}\alpha_1 y(0)+\alpha_2 y'(0)=0\\ \beta_1 y(L)+\beta_2 y'(L)=0\end{cases}$

where $\alpha_1^2+\alpha_2^2>0$ and $\beta_1^2+\beta_2^2>0$

The eigenvalues $\lambda_1\cdots$ are real and can be ordered $\lambda_1<\lambda_2<\cdots$ and $\lambda_n\to\infty$ as $n\to\infty$

for each λ_n, the $y_n(x)$ has $n-1$ zeros in $(0,L)$, satisfy $\int_0^L y_m(x)y_n(x)dx=0$ for all $m\neq n$

if $f(x)$ and $f'(x)$ are piecewise continuous on $[0,L]$ the $f(x)=\sum_{n=1}^{\infty}C_n y_n(x) \Rightarrow C_n=\frac{\int_0^L f(x)y_n(x)dx}{\int_0^L y_n^2(x)dx}$

For the Regular Sturm-Liouville Problem

$(p(x)y')'+q(x)y+\lambda w(x)y=0$ $a<x<b$ $\begin{cases}\alpha_1 y(a)+\alpha_2 y'(a)=0\\ \beta_1 y(b)+\beta_2 y'(b)=0\end{cases}$

where $\alpha_1^2+\alpha_2^2>0$ and $\beta_1^2+\beta_2^2>0$ $p(x)>0$ and $w(x)>0$

for eigenfunctions $y_n(x)$ $\int_a^b y_m(x)y_n(x)w(x)dx=0$ for $\forall m\neq n$

if $f(x)$ and $f'(x)$ are piecewise continuous on $[a,b]$ then

$$f(x) = \sum_{n=1}^{\infty} C_n y_n(x) \Rightarrow C_n = \frac{\int_a^b f(x) y_n(x) w(x)\,dx}{\int_a^b y_n^2(x) w(x)\,dx}$$

Fourier cosine series of f is (on $[0,L]$)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \text{ for all } x \in [0,L]$$

where $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$ $n \geq 1$

Fourier sine series of f is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \text{ for } \forall x \in [0,L] \text{ where } b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,\ n \geq 1$$

Fourier series

suppose f and f' are piecewise cts on $[-L,L]$. Then f has Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ $n \geq 1$ $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$ $n \geq 1$

PDE examples

① $u_t = c^2 u_{xx}$ $0<x<L,\ 0<t$ $u(0,t)=0,\ u(L,t)=0$ $0<t$ $u(x,0)=f(x)$ $0\leq x\leq L$

Solution: $u(x,t) = X(x)T(t)$ Then $X(x)T'(t) = u_t = c^2 u_{xx} = c^2 X''(x)T(t)$

Thus $\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$ $u(0,t)=0$ and $u(L,t)=0$ $t>0 \Rightarrow X(0)=0, X(L)=0$

$$\Rightarrow \begin{cases} T'(t) = -\lambda c^2 T(t) \\ X''(x) + \lambda X(x) = 0 \text{ where } X(0)=0 \text{ and } X(L)=0 \end{cases}$$

eigenvalue $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and eigenfunction $X_n(x) = \sin\frac{n\pi x}{L}$

for 1st DE, $T_n(t) = b_n e^{-\lambda_n c^2 t} = b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}$

$$\Rightarrow u_n(x,t) = X_n(x)T_n(t) = b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}\sin\frac{n\pi x}{L} \quad n=1,2\cdots$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}\sin\frac{n\pi x}{L}$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \Rightarrow b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}\sin\frac{n\pi x}{L} \text{ with } b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Example 2

$u_t = c^2 u_{xx} \quad 0<x<L, \quad t>0 \;;\; u_x(0,t)=0,\; u_x(L,t)=0 \quad 0<t \;;\; u(x,0)=f(x) \quad 0\le x\le L$

$u(x,t)=X(x)T(t)$ Then $\frac{T'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda \Rightarrow \begin{cases} T'(t)=-c^2\lambda T(t) \\ X''(x)+\lambda X(x)=0 \end{cases}$ where $X'(0)=0 \quad X'(L)=0$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad X_n = \cos\frac{n\pi x}{L} \quad T_n(t) = a_n e^{-\lambda_n c^2 t} = a_n e^{-\left(\frac{n\pi c}{L}\right)^2 t}$

Then $u_n(x,t) = X_n(x)\,T_n(t) = a_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \cos\frac{n\pi x}{L} \quad n=0,1,2\cdots$

the general: $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi c}{L}\right)^2 t} \cos\frac{n\pi x}{L}$

$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$