Summary of LMM

Based on data

$$(y_{ij}; x_{1ij}, ..., x_{pij}; z_{1ij}, ..., z_{qij}) \equiv (y_{ij}; \boldsymbol{x_{ij}}; \boldsymbol{z_{ij}}),\ i = 1, ..., M, j = 1, ..., n_i,$$

the Laird-Ware form of the linear mixed model (LMM) is denoted as

$$y_{ij} = \beta_1 + \beta_2 x_{2ij} + ... + \beta_p x_{pij} + b_{1i} + b_{2i} z_{2ij} + ... + b_{qi} z_{qij} + \epsilon_{ij}, i = 1, ..., M, j = 1, ..., n_i, \quad (1)$$

where y_{ij} is the value of the response variable for the j-th of n_i observations in the i-th of M groups or clusters. Let

$$\boldsymbol{b_i} = \begin{pmatrix} b_{1i} \\ ... \\ b_{ki} \\ ... \\ b_{qi} \end{pmatrix}_{q\times 1}, \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ ... \\ \beta_l \\ ... \\ \beta_p \end{pmatrix}_{p\times 1}, \boldsymbol{\epsilon_i} = \begin{pmatrix} \epsilon_{i1} \\ ... \\ \epsilon_{ij} \\ ... \\ \epsilon_{in_i} \end{pmatrix}_{n_i\times 1}, \text{ and } \boldsymbol{y_i} = \begin{pmatrix} y_{i1} \\ ... \\ y_{ij} \\ ... \\ y_{in_i} \end{pmatrix}_{n_i\times 1}.$$

The Laird-Ware model in matrix notation is denoted as

$$\mathbf{y_i} = \mathbf{X_i}\boldsymbol{\beta} + \mathbf{Z_i}\mathbf{b_i} + \boldsymbol{\epsilon_i}\ ,\ i = 1, ..., M, \quad (2)$$

$$\text{where } \boldsymbol{X_i} = \begin{pmatrix} x_{i11} & x_{i12} & ... & x_{i1p} \\ ... & ... & ... & ... \\ x_{ij1} & x_{ij2} & ... & x_{ijp} \\ ... & ... & ... & ... \\ x_{in_i1} & x_{in_i2} & ... & x_{in_ip} \end{pmatrix}_{n_i\times p}, \text{ and } \boldsymbol{Z_i} = \begin{pmatrix} z_{i11} & z_{i12} & ... & z_{i1q} \\ ... & ... & ... & ... \\ z_{ij1} & z_{ij2} & ... & z_{ijq} \\ ... & ... & ... & ... \\ z_{in_i1} & z_{in_i2} & ... & z_{in_iq}, \end{pmatrix}_{n_i\times q}.$$

In model (2), $\boldsymbol{\beta}$ is a p-dimensional unknown vector of regression coefficient, representing *common* fixed effects of the covariate $\boldsymbol{X_i}$ on the response value. $\boldsymbol{b_i}$ is q-dimensional random vector which follows multivariate normal (Gaussian) distribution, and therefore represents random effects of the covariate $\boldsymbol{Z_i}$ on the response for the i-th group or cluster. The n_i-dimensional vector of model error $\boldsymbol{\epsilon_i}$ used to be Gaussian distribution also. Therefore the distributions of the n_i-dimensional response vector $\boldsymbol{y_i}$ will be determined jointly by $\boldsymbol{b_i}$ and $\boldsymbol{\epsilon_i}$.

Next we discuss the within (the group/ cluster i) and between (any two groups/ clusters i and i') structure under the Laird-Ware model.

- Within a group i (intra).
  - $\boldsymbol{b_i} \sim N_q(\mathbf{0}, \boldsymbol{\Psi})$. Within the group i, b_{ki} is the k-th random-effect coefficient. b_{ki} follows a normal distribution with mean 0 and varianve ψ_k^2.
    $Cov(b_{ki}, b_{k'i}) = \begin{cases} \psi_{kk'}, \text{ for } k \neq k' \\ \psi_k^2, \text{ for } k = k' \end{cases}$;

- $\boldsymbol{\epsilon_i} \sim N_{n_i}(0, \sigma^2 \boldsymbol{\Lambda_i})$. Within the group i, ϵ_{ij} is the j-th model error and follows a normal distribution with mean 0 and variance $\sigma^2 \lambda_{ijj}$,
  $Cov(\epsilon_{ji}, \epsilon_{j'i}) = \begin{cases} \sigma^2 \lambda_{ijj'}, \text{ for } j \neq j' \\ \sigma^2 \lambda_{ijj}, \text{ for } j = j' \end{cases}$;
- $\boldsymbol{b_i} \perp \boldsymbol{\epsilon_i}$. Here "$\perp$" denotes independence.

- Between groups i and i', $i \neq i'$ (inter)
  - Consider $\boldsymbol{y_i = X_i \beta + Z_i b_i + \epsilon_i}$ and $\boldsymbol{y_{i'} = X_{i'} \beta + Z_{i'} b_{i'} + \epsilon_{i'}}$.
  - $\boldsymbol{b_i} \perp \boldsymbol{b_{i'}} \Rightarrow Cov(\boldsymbol{b_i}, \boldsymbol{b_{i'}}) = 0$
  - $\boldsymbol{\epsilon} \perp \boldsymbol{\epsilon_{i'}} \Rightarrow Cov(\boldsymbol{\epsilon_i}, \boldsymbol{\epsilon_{i'}}) = 0$
  - $\boldsymbol{b_i} \perp \boldsymbol{\epsilon_i} \Rightarrow Cov(\boldsymbol{b_i}, \boldsymbol{\epsilon_i}) = 0$

- Let $\sum_{i=1}^{M} n_i \equiv N$.
  - $\boldsymbol{y_N} = \begin{pmatrix} \boldsymbol{y_1} \\ \cdots \\ \boldsymbol{y_i} \\ \cdots \\ \boldsymbol{y_M} \end{pmatrix}_{N \times 1}$, where $\boldsymbol{y_i} = \begin{pmatrix} y_{i1} \\ \cdots \\ y_{ij} \\ \cdots \\ y_{in_i} \end{pmatrix}_{n_i \times 1}$;
  - $\boldsymbol{X_N} = \begin{pmatrix} \boldsymbol{X_1} \\ \cdots \\ \boldsymbol{X_i} \\ \cdots \\ \boldsymbol{X_M} \end{pmatrix}_{N \times p}$, where $\boldsymbol{X_i} = \begin{pmatrix} x_{i11} & x_{i12} & \cdots & x_{i1p} \\ \cdots & \cdots & \cdots & \cdots \\ x_{ij1} & x_{ij2} & \cdots & x_{ijp} \\ \cdots & \cdots & \cdots & \cdots \\ x_{in_i1} & x_{in_i2} & \cdots & x_{in_ip} \end{pmatrix}_{n_i \times p}$;
  - $\boldsymbol{\epsilon_N} = \begin{pmatrix} \boldsymbol{\epsilon_1} \\ \cdots \\ \boldsymbol{\epsilon_i} \\ \cdots \\ \boldsymbol{\epsilon_M} \end{pmatrix}_{N \times 1}$, where $\boldsymbol{\epsilon_i} = \begin{pmatrix} \epsilon_{i1} \\ \cdots \\ \epsilon_{ij} \\ \cdots \\ \epsilon_{in_i} \end{pmatrix}_{n_i \times 1}$
  - $E(\boldsymbol{y_N}) = E(\boldsymbol{X_N \beta}) = \boldsymbol{X_N \beta}$
  - $Var(\boldsymbol{y_N}) = Var(c + \sum_{\boldsymbol{k=1}}^{\boldsymbol{q}} \boldsymbol{Z_{kij} b_{ki}} + \boldsymbol{\epsilon_{ij}}) = \cdots$