

AMA 2634

Introduction to Statistics

2024 Fall

Lecture 4

Probability

Probability can be defined as the chance of an event occurring. It can be used to quantify what the “odds” are that a specific event will occur. Some examples of how probability is used everyday would be weather forecasting, “75% chance of snow” or for setting insurance rates.



A **probability experiment** is a chance process that leads to well-defined results called outcomes.

An **outcome** is the result of a single trial of a probability experiment.

A **sample space** is the set of all possible outcomes of a probability experiment.

An **event** consists of outcomes.

Experiment	Sample Space
Toss a coin	Head, Tail
Roll a die	1, 2, 3, 4, 5, 6
Toss two coins	HH, HT, TH, TT

Example: Rolling Dice

Find the sample space for rolling two dice.

Die 1	Die 2					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Example: Gender of Children



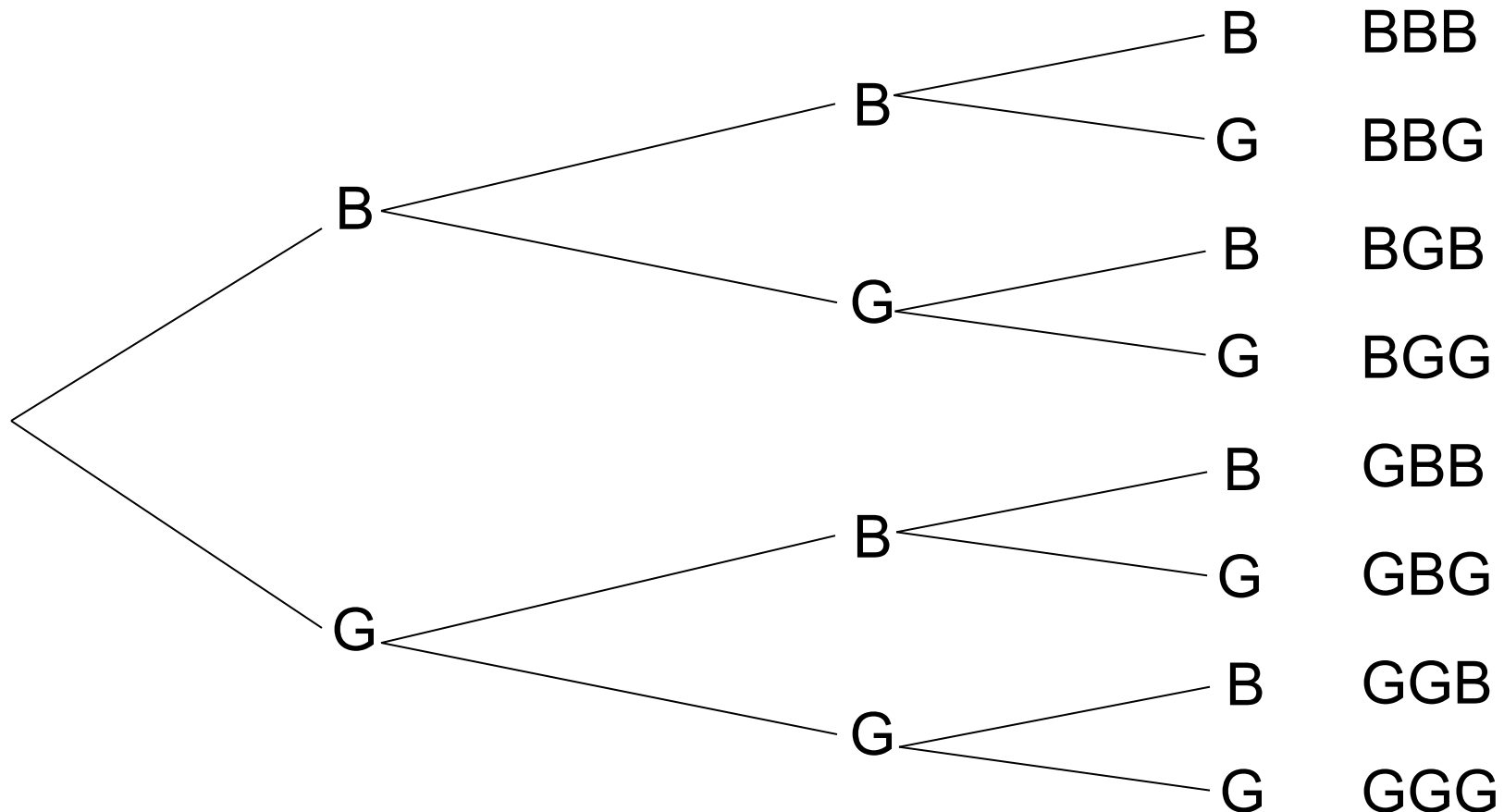
Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.

BBB BBG BGB BGG GBB GBG GGB GGG

Example: Gender of Children



Use a tree diagram to find the sample space for the gender of three children in a family.





There are three basic interpretations of probability:

Classical probability

Empirical probability

Subjective probability

Classical probability uses sample spaces to determine the numerical probability that an event will happen and assumes that all outcomes in the sample space are equally likely to occur.

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{\# of desired outcomes}}{\text{Total \# of possible outcomes}}$$

Example: Gender of Children



If a family has three children, find the probability that two of the three children are girls.

Sample Space:

BBB BBG BGB BGG GBB GBG GGB GGG

Three outcomes (BGG, GBG, GGB) have two girls.

The probability of having two of three children being girls is $\frac{3}{8}$.

Let E be an event, then $0 \leq P(E) \leq 1$.

If an event E cannot occur (i.e., the event contains no members in the sample space), its probability is 0.

If an event E is certain, then the probability of E is 1.

The sum of the probabilities of all the outcomes in the sample space is 1.

Exercise: Rolling a Die



When a single die is rolled, what is the probability of getting a number less than 7?

Since all outcomes—1, 2, 3, 4, 5, and 6—are less than 7, the probability is

$$P(\text{number less than 7}) = \frac{6}{6} = 1$$

The event of getting a number less than 7 is certain.



The *complement of an event E* , denoted by $\bar{E}$, is the set of outcomes in the sample space that are not included in the outcomes of event E .

$$P(E) = 1 - P(\bar{E})$$

Example: Finding Complements



Find the complement of each event.

Event	Complement of the Event
Rolling a die and getting a 4	Getting a 1, 2, 3, 5, or 6
Selecting a letter of the alphabet and getting a vowel	Getting a consonant (assume y is a consonant)
Selecting a month and getting a month that begins with a J	Getting February, March, April, May, August, September, October, November, or December
Selecting a day of the week and getting a weekday	Getting Saturday or Sunday

If the probability that a person lives in an industrialized country of the world is $\frac{1}{5}$, find the probability that a person does not live in an industrialized country.

$$\begin{aligned} P(\text{Not living in industrialized country}) \\ &= 1 - P(\text{living in industrialized country}) \\ &= 1 - \frac{1}{5} = \boxed{\frac{4}{5}} \end{aligned}$$

Empirical probability relies on actual experience to determine the likelihood of outcomes.

$$P(E) = \frac{f}{n} = \frac{\text{frequency of desired class}}{\text{Sum of all frequencies}}$$

Example: Blood Types



In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a. A person has type O blood.

Type	Frequency
A	22
B	5
AB	2
O	21
Total	50

$$\begin{aligned} P(O) &= \frac{f}{n} \\ &= \frac{21}{50} \end{aligned}$$

Example: Blood Types



In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

b. A person has type A or type B blood.

Type	Frequency
A	22
B	5
AB	2
O	21
Total	50

$$\begin{aligned} P(A \text{ or } B) &= \frac{22}{50} + \frac{5}{50} \\ &= \frac{27}{50} \end{aligned}$$

Example: Blood Types



In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

c. A person has neither type A nor type O blood.

Type	Frequency
A	22
B	5
AB	2
O	21
Total	50

$$\begin{aligned} P(\text{neither A nor O}) &= \frac{5}{50} + \frac{2}{50} \\ &= \frac{7}{50} \end{aligned}$$

Example: Blood Types



In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

d. A person does not have type AB blood.

Type	Frequency
A	22
B	5
AB	2
O	21
Total	50

$$\begin{aligned} P(\text{not AB}) \\ &= 1 - P(\text{AB}) \\ &= 1 - \frac{2}{50} = \frac{48}{50} = \frac{24}{25} \end{aligned}$$



Subjective probability uses a probability value based on an educated guess or estimate, employing opinions and inexact information.

Examples: weather forecasting, predicting outcomes of sporting events

Two events are **mutually exclusive events** if they cannot occur at the same time (i.e., they have no outcomes in common)

Addition Rules

$$P(A \text{ or } B) = P(A) + P(B) \quad \text{Mutually Exclusive}$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \quad \text{Not M. E.}$$

Example: Rolling a Die



Determine which events are mutually exclusive and which are not, when a single die is rolled.

a. Getting an odd number and getting an even number

Getting an odd number: 1, 3, or 5

Getting an even number: 2, 4, or 6

Mutually Exclusive

Example: Rolling a Die



Determine which events are mutually exclusive and which are not, when a single die is rolled.

b. Getting a 3 and getting an odd number

Getting a 3: 3

Getting an odd number: 1, 3, or 5

Not Mutually Exclusive

Determine which events are mutually exclusive and which are not, when a single die is rolled.

c. Getting an odd number and getting a number less than 4

Getting an odd number: 1, 3, or 5

Getting a number less than 4: 1, 2, or 3

Not Mutually Exclusive

Determine which events are mutually exclusive and which are not, when a single die is rolled.

d. Getting a number greater than 4 and getting a number less than 4

Getting a number greater than 4: 5 or 6

Getting a number less than 4: 1, 2, or 3

Mutually Exclusive

Example: R&D Employees



The corporate research and development centers for three local companies have the following number of employees:

U.S. Steel 110

Alcoa 750

Bayer Material Science 250

If a research employee is selected at random, find the probability that the employee is employed by U.S. Steel or Alcoa.

$$\begin{aligned} P(\text{U.S. Steel or Alcoa}) &= P(\text{U.S. Steel}) + P(\text{Alcoa}) \\ &= \frac{110}{1110} + \frac{750}{1110} = \frac{860}{1110} = \frac{86}{111} \end{aligned}$$

Example: Medical Staff



In a hospital unit there are 8 nurses and 5 doctors;
7 nurses and 3 doctors are females.

If a staff person is selected, find the probability that
the subject is a nurse or a male.

Staff	Females	Males	Total
Nurses	7	1	8
doctors	3	2	5
Total	<u>10</u>	<u>3</u>	<u>13</u>

$$\begin{aligned}P(\text{Nurse or Male}) &= P(\text{Nurse}) + P(\text{Male}) - P(\text{Male Nurse}) \\&= \frac{8}{13} + \frac{3}{13} - \frac{1}{13} = \boxed{\frac{10}{13}}\end{aligned}$$

- Two events A and B are **independent events** if the fact that A occurs does not affect the probability of B occurring.

Multiplication Rules

$$P(A \text{ and } B) = P(A) \cdot P(B) \quad \text{Independent}$$

$$P(A \text{ and } B) = P(A) \cdot P(B|A) \quad \text{Dependent}$$

Example: Tossing a Coin



A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Independent Events

$$\begin{aligned} P(\text{Head and } 4) &= P(\text{Head}) \cdot P(4) \\ &= \frac{1}{2} \cdot \frac{1}{6} = \boxed{\frac{1}{12}} \end{aligned}$$

This problem could be solved using sample space.

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6

A Harris poll found that 46% of Americans say they suffer great stress at least once a week. If three people are selected at random, find the probability that all three will say that they suffer great stress at least once a week.

Independent Events

$$\begin{aligned}P(S \text{ and } S \text{ and } S) &= P(S) \cdot P(S) \cdot P(S) \\&= (0.46)(0.46)(0.46) \\&= \boxed{0.097}\end{aligned}$$

At a university in western Pennsylvania, there were 5 burglaries reported in 2003, 16 in 2004, and 32 in 2005. If a researcher wishes to select at random two burglaries to further investigate, find the probability that both will have occurred in 2004.

Dependent Events

$$\begin{aligned} P(C_1 \text{ and } C_2) &= P(C_1) \cdot P(C_2 | C_1) \\ &= \frac{16}{53} \cdot \frac{15}{52} = \frac{60}{689} \end{aligned}$$

■ **Conditional probability** is the probability that the second event B occurs given that the first event A has occurred.

Conditional Probability

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example: Parking Tickets



The probability that Sam parks in a no-parking zone *and gets a parking ticket is 0.06*, and the probability that Sam cannot find a legal parking space and has to park in the no-parking zone is 0.20. On Tuesday, Sam arrives at school and has to park in a no-parking zone. Find the probability that he will get a parking ticket.

N = parking in a no-parking zone

T = getting a ticket

$$P(T | N) = \frac{P(N \text{ and } T)}{P(N)} = \frac{0.06}{0.20} = \boxed{0.30}$$

Example: Women in the Military



A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown.

Gender	Yes	No	Total
Male	32	18	50
Female	<u>8</u>	<u>42</u>	<u>50</u>
Total	40	60	100

Example: Women in the Military



- a. Find the probability that the respondent answered yes (Y), given that the respondent was a female (F).

Gender	Yes	No	Total
Male	32	18	50
Female	<u>8</u>	<u>42</u>	<u>50</u>
Total	40	60	100

$$P(Y|F) = \frac{P(F \text{ and } Y)}{P(F)} = \frac{\frac{8}{100}}{\frac{50}{100}} = \frac{8}{50} = \boxed{\frac{4}{25}}$$

Example: Women in the Military



b. Find the probability that the respondent was a male (M), given that the respondent answered no (N).

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

$$P(M|N) = \frac{P(N \text{ and } M)}{P(N)} = \frac{\frac{18}{100}}{\frac{60}{100}} = \frac{18}{60} = \boxed{\frac{3}{10}}$$

Example: Bow Ties



The Neckware Association of America reported that 3% of ties sold in the United States are bow ties (B). If 4 customers who purchased a tie are randomly selected, find the probability that at least 1 purchased a bow tie.

$$P(B) = 0.03, P(\bar{B}) = 1 - 0.03 = 0.97$$

$$\begin{aligned} P(\text{no bow ties}) &= P(\bar{B}) \cdot P(\bar{B}) \cdot P(\bar{B}) \cdot P(\bar{B}) \\ &= (0.97)(0.97)(0.97)(0.97) = 0.885 \end{aligned}$$

$$\begin{aligned} P(\text{at least 1 bow tie}) &= 1 - P(\text{no bow ties}) \\ &= 1 - 0.885 = \boxed{0.115} \end{aligned}$$

- The **fundamental counting rule** is also called the **multiplication of choices**.
- In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3 , and so forth, the total number of possibilities of the sequence will be

$$k_1 \cdot k_2 \cdot k_3 \cdot \cdots k_n$$

Example: Paint Colors



A paint manufacturer wishes to manufacture several different paints. The categories include

Color: red, blue, white, black, green, brown, yellow

Type: latex, oil

Texture: flat, semigloss, high gloss

Use: outdoor, indoor

How many different kinds of paint can be made if you can select one color, one type, one texture, and one use?

$$\begin{array}{ccccccc} \left(\begin{array}{c} \# \text{ of} \\ \text{colors} \end{array} \right) & \left(\begin{array}{c} \# \text{ of} \\ \text{types} \end{array} \right) & \left(\begin{array}{c} \# \text{ of} \\ \text{textures} \end{array} \right) & \left(\begin{array}{c} \# \text{ of} \\ \text{uses} \end{array} \right) \\ 7 & \cdot & 2 & \cdot & 3 & \cdot & 2 \end{array}$$

84 different kinds of paint

■ **Factorial** is the product of all the positive numbers from 1 to a number.

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

$$0! = 1$$

■ **Permutation** is an arrangement of objects in a specific order. Order matters.

$${}_nP_r = \frac{n!}{(n-r)!} = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ items}}$$

■ **Combination** is a grouping of objects.
Order does not matter.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

$$= \frac{{}_nP_r}{r!}$$

Example: Business Location



Suppose a business owner has a choice of 5 locations in which to establish her business. She decides to rank each location according to certain criteria, such as price of the store and parking facilities. How many different ways can she rank the 5 locations?

$$\begin{pmatrix} \text{first} \\ \text{choice} \end{pmatrix} \begin{pmatrix} \text{second} \\ \text{choice} \end{pmatrix} \begin{pmatrix} \text{third} \\ \text{choice} \end{pmatrix} \begin{pmatrix} \text{fourth} \\ \text{choice} \end{pmatrix} \begin{pmatrix} \text{fifth} \\ \text{choice} \end{pmatrix}$$
$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

120 different ways to rank the locations
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Using factorials, $5! = 120$.

Using permutations, ${}_5P_5 = 120$.

Example: Business Location



Suppose the business owner in Example 4–42 wishes to rank only the top 3 of the 5 locations. How many different ways can she rank them?

$$\begin{array}{ccccc} \left(\begin{array}{c} \text{first} \\ \text{choice} \end{array} \right) & \left(\begin{array}{c} \text{second} \\ \text{choice} \end{array} \right) & \left(\begin{array}{c} \text{third} \\ \text{choice} \end{array} \right) \\ 5 & \cdot & 4 & \cdot & 3 \end{array}$$

60 different ways to rank the locations

Using permutations, ${}_5P_3 = 60$.

Example: Television Ads



The advertising director for a television show has 7 ads to use on the program.

If she selects 1 of them for the opening of the show, 1 for the middle of the show, and 1 for the ending of the show, how many possible ways can this be accomplished?

Since order is important, the solution is

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$$

Hence, there would be 210 ways to show 3 ads.

Example: School Musical Plays



A school musical director can select 2 musical plays to present next year. One will be presented in the fall, and one will be presented in the spring. If she has 9 to pick from, how many different possibilities are there?

Order matters, so we will use permutations.

$${}_9P_2 = \frac{9!}{7!} = 72 \quad \text{or} \quad {}_9P_2 = \underbrace{9 \cdot 8}_2 = 72$$

Example: Book Reviews



A newspaper editor has received 8 books to review. He decides that he can use 3 reviews in his newspaper. How many different ways can these 3 reviews be selected?

The placement in the newspaper is not mentioned, so order does not matter. We will use combinations.

$${}_8C_3 = \frac{8!}{5!3!} = 8! / (5!3!) = 56$$

$$\text{or } {}_8C_3 = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 56 \quad \text{or} \quad {}_8C_3 = \frac{{}_8P_3}{3!} = 56$$

Example 4-49: Committee Selection



In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

There are not separate roles listed for each committee member, so order does not matter. We will use combinations.

$$\text{Women: } {}_7C_3 = \frac{7!}{4!3!} = 35, \quad \text{Men: } {}_5C_2 = \frac{5!}{3!2!} = 10$$

There are $35 \cdot 10 = 350$ different possibilities.



The counting rules can be combined with the probability rules in this chapter to solve many types of probability problems.

By using the fundamental counting rule, the permutation rules, and the combination rule, you can compute the probability of outcomes of many experiments, such as getting a full house when 5 cards are dealt or selecting a committee of 3 women and 2 men from a club consisting of 10 women and 10 men.

A store has 6 *TV Graphic* magazines and 8 *Newstime* magazines on the counter. If two customers purchased a magazine, find the probability that one of each magazine was purchased.

TV Graphic: One magazine of the 6 magazines

Newstime: One magazine of the 8 magazines

Total: Two magazines of the 14 magazines

$$\frac{{}_6C_1 \cdot {}_8C_1}{{}_{14}C_2} = \frac{6 \cdot 8}{91} = \boxed{\frac{48}{91}}$$

Example: Combination Lock



A combination lock consists of the 26 letters of the alphabet. If a 3-letter combination is needed, find the probability that the combination will consist of the letters ABC in that order. The same letter can be used more than once. (*Note: A combination lock is really a permutation lock.*)

There are $26 \cdot 26 \cdot 26 = 17,576$ possible combinations. The letters ABC in order create one combination.

$$P(ABC) = \frac{1}{17,576}$$

Rule of Total Probability

Let E be an event that is a subset of a sample space S . Let $A_1, A_2, \dots, A_n$ be partitions of the sample space S . Then

$$P(E) = P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + \cdots + P(A_n) \cdot P(E|A_n)$$



- In a particular town
 - 30% of the voters are Republican
 - 30% of the voters are Democrats
 - 40% of the voters are independents
- This is a partition of the voters into three sets
 - There are no voters that are in two sets (disjoint)
 - All voters are in one of the sets (covers all of S)
- For a particular issue
 - 90% of the Republicans favor it
 - 60% of the Democrats favor it
 - 70% of the independents favor it
- These are the conditional probabilities
 - $E = \{\text{favor the issue}\}$
 - The above probabilities are $P(E \mid \text{political party})$

- The total proportion of votes who favor the issue

$$0.3 \cdot 0.9 + 0.3 \cdot 0.6 + 0.4 \cdot 0.7 = 0.73$$

- So 73% of the voters favor this issue

Bayes's Rule

Let $A_1, A_2, \dots, A_n$ be partitions of a sample space S . Then, for any event E for which $P(E) > 0$, the probability of event A_i for $i = 1, 2, \dots, n$, given the event E , is

$$\begin{aligned} P(A_i|E) &= \frac{P(A_i) \cdot P(E|A_i)}{P(E)} \\ &= \frac{P(A_i) \cdot P(E|A_i)}{P(A_1) \cdot P(E|A_1) + P(A_2) \cdot P(E|A_2) + \cdots + P(A_n) \cdot P(E|A_n)} \end{aligned}$$